Analytical Physics 1B Lecture 1:
Rotation of Rigid Bodies

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Friday, January 25th, 2019
**SOME ADVICE**

- This class is cumulative
  - You will be in bad shape if you can’t remember anything from last semester
  - Topics are also change a lot (more so than last semester); important to stay on top of the material each week

- In my own experience, most of my learning came from doing the problem sets
  - My advice: take the homework and workshops seriously
    - homework questions will appear (in modified form) on exams, so you’ll be rewarded for working through it carefully
  - To get the most out of lecture, do the reading **before class**
    - It will be a lot harder if you try to learn the material in the lecture directly
    - iClicker questions will re-appear on the quizzes, and maybe even on the final exam


RIGHT-HAND RULE (REVIEW)

\[ \mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]
**Cross Product (Review)**

\[ \mathbf{A} \times \mathbf{B} \equiv |\mathbf{A}| |\mathbf{B}| \sin \theta \]

\[ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \]

\[ \mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \]

\[ \mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \]

\[ \mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} \]

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]
RADIANS

• Radians are the proper units that satisfy the equation $s = r\theta$ for a circle
  • $s = \text{arc length}$
  • $r = \text{radius}$

An angle $\theta$ in radians is the ratio of the arc length $s$ to the radius $r$. 

(b) $s = r\theta$
**Angular Velocity**

\[
\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \quad \text{(definition of angular velocity)}
\]

- Define angular velocity as the change in **angular distance** with respect to time
  - common units are radians/s (or rad/s)
  - Since \(2\pi\) radians = 1 revolution, 1 rev/s = 1 Hertz = \(2\pi\) rad/s

\[
\omega = 2\pi f \quad \text{and} \quad f = \frac{\omega}{2\pi}
\]

- \(\omega\)=angular velocity [rad/s]
- \(f\)=frequency [cycles/s]
• Angular velocity is a **vector** so it has a **direction**
  • it is often conventional to pick the z axis in problems as the direction for angular velocity

• What is the difference between angular and linear velocity?

\[ \mathbf{v} = r \mathbf{\omega} \] (relationship between linear and angular speeds)
CALVIN AND HOBBISS

PLAYING A RECORD? I'LL SHOW YOU SOMETHING INTERESTING.

COMPARE A POINT ON THE LABEL WITH A POINT ON THE RECORD'S OUTER EDGE. THEY BOTH MAKE A COMPLETE CIRCLE IN THE SAME AMOUNT OF TIME, RIGHT?

YEAH...

BUT THE POINT ON THE RECORD'S EDGE HAS TO MAKE A BIGGER CIRCLE IN THE SAME TIME, SO IT GOES FASTER.

SEE, TWO POINTS ON ONE DISK MOVE AT TWO SPEEDS, EVEN THOUGH THEY BOTH MAKE THE SAME REVOLUTIONS PER MINUTE!
**Angular Acceleration**

- Define angular acceleration as the change in angular velocity with respect to time

\[ \alpha_z = \lim_{\Delta t \to 0} \frac{\Delta \omega_z}{\Delta t} = \frac{d\omega_z}{dt} \]  
  (definition of angular acceleration)

- Angular acceleration is also a vector, so it too has a direction
  - notice that the angular acceleration does not have to be in the same direction as the angular velocity

- Similarly, it can be related to the linear acceleration by the equation:

\[ a_{\text{tan}} = \frac{dv}{dt} = r \frac{d\omega}{dt} = r \alpha \]  
  (tangential acceleration of a point on a rotating body)
CENTRIPETAL ACCELERATION

• As you learned last semester, the centripetal acceleration is $a_{\text{rad}} = \frac{v^2}{r}$
  • we can now re-express this in terms of the angular velocity

\[
a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r
\]
(centripetal acceleration of a point on a rotating body)

• Don’t confuse accelerations!
  • For point P in the figure, which direction is the angular acceleration pointing? The centripetal acceleration? The linear acceleration?

Physics 124 – Rotation of Rigid Bodies
### Table 9.1 Comparison of Linear and Angular Motion with Constant Acceleration

<table>
<thead>
<tr>
<th>Straight-Line Motion with Constant Linear Acceleration</th>
<th>Fixed-Axis Rotation with Constant Angular Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_x = \text{constant} )</td>
<td>( \alpha_z = \text{constant} )</td>
</tr>
<tr>
<td>( v_x = v_{0x} + a_x t )</td>
<td>( \omega_z = \omega_{0z} + \alpha_z t )</td>
</tr>
<tr>
<td>( x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 )</td>
<td>( \theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2 )</td>
</tr>
<tr>
<td>( v_x^2 = v_{0x}^2 + 2a_x(x - x_0) )</td>
<td>( \omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) )</td>
</tr>
<tr>
<td>( x - x_0 = \frac{1}{2}(v_{0x} + v_x)t )</td>
<td>( \theta - \theta_0 = \frac{1}{2} (\omega_{0z} + \omega_z)t )</td>
</tr>
</tbody>
</table>
ENERGY IN ROTATIONAL MOTION

• In a “rigid body” the distance between any two given points of the body remains constant in time regardless of external forces exerted on it
  • This is an idealization we use commonly in physics
  • Not all bodies are rigid, but those that we will study in this class are

• A rotating rigid body (even one with net velocity=0) consists of mass in motion, so it has kinetic energy. We can express this kinetic energy in terms of the body’s angular speed and a new quantity, called moment of inertia, that depends on the body’s mass and how the mass is distributed.
ENERGY IN ROTATIONAL MOTION

• Consider a rigid body rotating about a fixed axis
  • The body is made up of a large number (infinite?) of points each moving with respect to that axis
  • The rotational kinetic energy of body is

\[ K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \cdots = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 \]

• Collecting terms we can pull out a factor of \( \omega^2/2 \):

\[ K = \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + \cdots) \omega^2 = \frac{1}{2} \left( \sum_i m_i r_i^2 \right) \omega^2 \]

• Therefore:

\[ I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \quad \text{(definition of moment of inertia)} \]

\[ K = \frac{1}{2} I \omega^2 \quad \text{(rotational kinetic energy of a rigid body)} \]
Moments of Inertia

- Moments of inertia look like a geometrical factor (a number) times the mass of the body times a distance squared.

<table>
<thead>
<tr>
<th>Table 9.2</th>
<th>Moments of Inertia of Various Bodies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Slender rod, axis through center</td>
<td>$I = \frac{1}{12} ML^2$</td>
</tr>
<tr>
<td>(b) Slender rod, axis through one end</td>
<td>$I = \frac{1}{3} ML^2$</td>
</tr>
<tr>
<td>(c) Rectangular plate, axis through center</td>
<td>$I = \frac{1}{12} M(a^2 + b^2)$</td>
</tr>
<tr>
<td>(d) Thin rectangular plate, axis along edge</td>
<td>$I = \frac{1}{3} Ma^2$</td>
</tr>
<tr>
<td>(e) Hollow cylinder</td>
<td>$I = \frac{1}{2} M(R_1^2 + R_2^2)$</td>
</tr>
<tr>
<td>(f) Solid cylinder</td>
<td>$I = \frac{1}{2} MR^2$</td>
</tr>
<tr>
<td>(g) Thin-walled hollow cylinder</td>
<td>$I = MR^2$</td>
</tr>
<tr>
<td>(h) Solid sphere</td>
<td>$I = \frac{2}{5} MR^2$</td>
</tr>
<tr>
<td>(i) Thin-walled hollow sphere</td>
<td>$I = \frac{2}{3} MR^2$</td>
</tr>
</tbody>
</table>

- Notice that they also depend on the axis of rotation.
Three flat objects (circular ring, circular disc, and square loop) have the same mass $M$ and the same outer dimension (circular objects have diameters of $2R$ and the square loop has sides of $2R$). The small circle at the center of each figure represents the axis of rotation for these objects. This axis of rotation passes through the center of mass and is perpendicular to the plane of the objects.

Rank the moments of inertia of these objects about the axis of rotation.

1. $A=B=C$
2. $C<A<B$
3. $B<A=C$
4. $A<B<C$
5. $B<A<C$
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Rolling Without Slipping

- The condition for rolling without slipping is $v_{cm} = R\omega$.

Translation of center of mass: velocity $\vec{v}_{cm}$

Rotation around center of mass: for rolling without slipping, speed at rim = $v_{cm}$

Combined motion

Wheel is instantaneously at rest where it contacts the ground.
A circular ring and circular disk with the same mass and radius start off with the same velocity at the base of a ramp with height $h$. The surface of the ramp has friction, so the ring and disk roll up the ramp without slipping.

Rank the height that each object climbs

1. $A=B$
2. $A<B$
3. $A>B$
4. Not enough information
A circular ring and circular disk with the same mass and radius start off with the same velocity at the base of a ramp with height h. The surface of the ramp has friction, so the ring and disk roll up the ramp without slipping.

Rank the height that each object climbs

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2. A<B
3. A>B
4. Not enough information
A circular ring and circular disk with the same mass and radius start off with the same velocity at the base of a ramp with height $h$. The surface of the ramp is completely frictionless, so the ring and disk roll up the ramp *with slipping*.

Rank the height that each object climbs

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A circular ring and circular disk with the same mass and radius start off with the same velocity at the base of a ramp with height $h$. The surface of the ramp is completely frictionless, so the ring and disk roll up the ramp with slipping.

Rank the height that each object climbs

1. $A=B$
2. $A<B$
3. $A>B$
4. Not enough information
**Parallel Axis Theorem**

- There is a simple relationship between the moment of inertia $I_{cm}$ of a body of mass $M$ about an axis through its center of mass and the moment of inertia $I_P$ about any other axis parallel to the original one but displaced from it by a distance $d$.

\[ I_P = I_{cm} + Md^2 \]  
(parallel-axis theorem)