Rutgers University
Department of Physics & Astronomy

Honors Analytical Physics I

Lecture 9
Midterm II

- Sunday November 18th 12:00pm in the Physics Lecture Hall

- Forces I and II i.e. Chapters 5 and 6 in textbook, Lectures 5,6,7 and first half of 8

- No energy and work
7. Kinetic energy and work

Energy

- Kinetic energy: associated to motion
- Potential energy: associated to the capacity of generating motion
• **Kinetic Energy**: energy associated to the motion of an object

\[ K = \frac{1}{2}mv^2 \]

• **Work-Kinetic Energy Theorem**

\[
\left( \text{change in the kinetic energy of a particle} \right) = \left( \text{net work done on the particle} \right).
\]

\[ \Delta K = W \quad K_f = K_i + W \]

\( W = \) net work done by all forces acting on the system
• For **straight** path and **constant** net force

\[ W = \vec{F}_{\text{net}} \cdot \vec{d} \]

where

\[ \vec{d} = \Delta \vec{r} \]

is the displacement.

• For **curved path** and **variable** net force:

Infinitesimal work : \[ \delta W = \vec{F}_{\text{net}} \cdot d\vec{s} \]

Total work : \[ W = \int_{\text{trajectory}} \vec{F}_{\text{net}} \cdot d\vec{s} \]
Kinetic-energy theorem in infinitesimal form

\[ dK = \delta W \]

instantaneous relation at any point along the trajectory.

\[ \Delta K = \int_{\text{trajectory}} \vec{F}_{\text{net}} \cdot d\vec{s} \]

for the entire motion.
Example

A ball of mass $m$ attached to string is launched with initial speed $v_0$ on a circular trajectory on horizontal plane. The friction force between the ball and the surface has constant magnitude $f_k$. What is the speed of the ball after travelling a distance $s$ along the circle.
Infinitesimal displacement

\[ \vec{ds} = \vec{v} dt \]

Note that \( \vec{T} \perp \vec{ds} \) while \( \vec{f}_k \) makes an angle \( \theta = 180^\circ \) with \( \vec{ds} \).

Infinitesimal work:

\[ dW_{f_k} = \vec{f}_k \cdot \vec{ds} = -f_k ds \]

Total work:

\[ W = \int dW = -f_k s \]

Work-energy theorem:

\[ \frac{mv^2}{2} - \frac{mv_0^2}{2} = -f_k s \]

\[ v = \sqrt{v_0^2 - \frac{2f_k s}{m}} \]
• **Work done by the gravitational force**

Freely falling object moving downwards:

\[ W = \vec{F}_g \cdot \vec{d} = mgd \cos 0^\circ = mgd > 0 \]

\[
K_1 = \frac{mv_1^2}{2} \quad K_2 = \frac{mv_2^2}{2}
\]

\[ K_2 - K_1 = mgd > 0 \]
Freely falling object moving upwards:

\[ W = \vec{F}_g \cdot \vec{d} = mgd \cos 180^\circ = -mgd < 0 \]

\[ K_1 = \frac{mv_1^2}{2} \quad K_2 = \frac{mv_2^2}{2} \]

\[ K_2 - K_1 = -mgd < 0 \]
An object of mass $m$ is launched with initial speed $v_0$ along an inclined plane making an angle $\theta = 45^\circ$ with the horizontal. The kinetic friction coefficient between the object and the plane is $\mu_k = 0.5$. Let $W_{f_k}$ be the total work done by the friction force until it stops. Which of the following statements is false?

A) $W_{f_k} < 0$

B) $W_{f_k} = -mv_0^2/2$

C) $|W_{f_k}| < mv_0^2/2$
Answer

An object of mass $m$ is launched with initial speed $v_0$ along an inclined plane making an angle $\theta = 45^\circ$ with the horizontal. The kinetic friction coefficient between the object and the plane is $\mu_k = 0.5$. Let $W_{fk}$ be the total work done by the friction force until it stops. Which of the following statements is false?

A) $W_{fk} < 0$

B) $W_{fk} = -mv_0^2/2$

C) $|W_{fk}| < mv_0^2/2$
\[ \vec{F}_{\text{net}} = m\vec{a} \quad (F_{\text{net}})_x = ma_x \quad (F_{\text{net}})_y = ma_y = 0 \]

\[ \vec{F}_{\text{net}} = \vec{F}_g + \vec{F}_N + \vec{f}_k \]

\[ (F_{\text{net}})_x = -mg\sin\theta - f_k \quad (F_{\text{net}})_y = F_N - mg\cos\theta = 0 \]

\[ F_N = mg\cos\theta \quad f_k = \mu_k mg\cos\theta \]
Work done by kinetic friction:
\[ W_{f_k} = \vec{f}_k \cdot \vec{d} = -f_k d = -\mu_k mgd \cos \theta \]

Work done by gravitational force:
\[ W_{Fg} = \vec{F}_g \cdot \vec{d} = -mgh = -mgd \sin \theta \]

Work done by normal force
\[ \vec{F}_N \cdot \vec{d} = 0 \]
Total work:
\[ W = W_{f_k} + W_{F_g} = -mgd (\sin \theta + \mu_k \cos \theta) \]

Work-Kinetic energy theorem:
\[ W = \Delta K = 0 - K_i = -\frac{mv_0^2}{2} \]

Work done by kinetic friction:
\[ \frac{W_{f_k}}{|W|} = -\frac{\mu_k \cos \theta}{\sin \theta + \mu_k \cos \theta} = \frac{1}{3} \]
\[ W_{f_k} = -\frac{mv_0^2}{6} \]
• **Work done by a spring force**

![Diagram of a spring and displaced block](image)

- **Hooke’s Law**
  
  \[ \vec{F}_s = -k\vec{d} \]

- always **opposed** to displacement (restoring force)

- \( k > 0 \) spring constant
\[ W_s = \int_{x_i}^{x_f} F_x \, dx \]
\[ = \int_{x_i}^{x_f} -kx \, dx \]
\[ = (-k) \int_{x_i}^{x_f} x \, dx \]
\[ = (-k/2) \left( x_f^2 - x_i^2 \right) \]
\[ W_s = \frac{1}{2} kx_i^2 - \frac{1}{2} kx_f^2 \]
• Example: an object of mass $m$ slides across a horizontal frictionless surface with speed $v$. It then runs into and compresses a spring of spring constant $k$. When the object is momentarily stopped by the spring, by what distance $d$ is the spring compressed?
Total work done by the spring force:

\[ W_s = \frac{kx_i^2}{2} - \frac{kx_f^2}{2} = -\frac{kd^2}{2} \]

Work-kinetic energy theorem

\[ W_s = K_f - K_i = -\frac{mv^2}{2} \]

\[ \frac{mv^2}{2} = \frac{kd^2}{2} \Rightarrow d = v\sqrt{\frac{m}{k}} \]
• **Power**: time rate at which work is done by a force.

If a force does an amount of work \( W \) in an amount of time \( \Delta t \), the **average power** during that time interval is:

\[
P_{\text{average}} = \frac{W}{\Delta t}
\]

The **instantaneous power** \( P \) is the instantaneous time rate of doing work

\[
P = \frac{\delta W}{dt} \quad \delta W = \vec{F} \cdot d\vec{s} = \vec{F} \cdot \vec{v} dt
\]

\[
P = \vec{F} \cdot \vec{v}
\]
• Units: Watt

\[ 1 \text{ Watt} = 1 \text{ W} = 1 \text{ J/s} \]
8. Potential energy. Conservation of energy

- **Potential energy**: energy associated with the configuration of a system of objects that exert forces on one another.

- can be converted into **kinetic energy** by allowing the system to evolve freely
Gravitational potential energy

Elastic potential energy

A ball at the top of a hill has potential energy but no kinetic energy.

A ball rolling down a hill has both kinetic and potential energy.

A ball at the bottom of the hill which is not rolling has neither potential nor kinetic energy.

A ball which has reached the bottom of the hill but is still rolling has kinetic energy but no potential energy.
• Work and potential energy

First part of motion:

\[ W_{Fg} = \Delta K < 0, \quad K \downarrow \]
energy transferred from kinetic energy to gravitational potential energy.

Second part of motion:

\[ W_{Fg} = \Delta K > 0, \quad K \uparrow \]
energy transferred from gravitational potential energy to kinetic energy.
First part of motion:

\[ W_{Fs} = \Delta K < 0, \quad K \downarrow \]
energy transferred from kinetic energy to elastic potential energy.

Second part of motion:

\[ W_{Fs} = \Delta K > 0, \quad K \uparrow \]
energy transferred from elastic potential energy to kinetic energy.
First part of motion:

\[ W_{Fg} = mg(y_0 - y_{\text{max}}) \]

\[ \Delta K = -\frac{mv_0^2}{2} \]

Constant acceleration model:

\[ \frac{mv_0^2}{2} = mg(y_{\text{max}} - y_0) \]

Second part of motion:

\[ W_{Fg} = mg(y_{\text{max}} - y_0) \]

\[ \Delta K = \frac{mv^2}{2} \]

Constant acceleration model:

\[ \frac{mv^2}{2} = mg(y_{\text{max}} - y_0) \]
First part of motion:

\[ W_{Fs} = -\frac{kx_{\text{max}}^2}{2} \]

\[ W_{Fs} = \Delta K = -\frac{mv_0^2}{2} \]

\[ \frac{mv_0^2}{2} = \frac{kx_{\text{max}}^2}{2} \]

Second part of motion:

\[ W_{Fs} = \frac{kx_{\text{max}}^2}{2} \]

\[ W_{Fs} = \Delta K = \frac{mv^2}{2} \]

\[ \frac{mv^2}{2} = \frac{kx_{\text{max}}^2}{2} \]
• Note that in both examples examples

\[ W_{1\text{-st part}} = -W_{2\text{-nd part}} \]

Gravitational force:

\[ \Delta (mgy) = -W_{Fg} \quad K + mgy = \text{constant} \]

Elastic force:

\[ \Delta (kx^2/2) = -W_s \quad K + \frac{kx^2}{2} = \text{constant} \]
Naturally led to:

- **Gravitational potential energy:**
  \[ U_g = mgy \]

- **Elastic potential energy:**
  \[ U_s = \frac{kx^2}{2} \]

- **Energy conservation:**
  \[ K + U_g = \text{constant} \quad K + U_s = \text{constant} \]
Which of the following statements is true?

A) $W_{Fg}^{(a)} > W_{Fg}^{(b)}$

B) $W_{Fg}^{(a)} < W_{Fg}^{(b)}$

C) $W_{Fg}^{(a)} = W_{Fg}^{(b)}$

D) none of the above.
Answer

Which of the following statements is true?

A) \( W_{Fg}^{(a)} > W_{Fg}^{(b)} \)

B) \( W_{Fg}^{(a)} < W_{Fg}^{(b)} \)

C) \( W_{Fg}^{(a)} = W_{Fg}^{(b)} \)

D) none of the above.
\[ W = \vec{F}_g \cdot \vec{d} = F_{gx} \Delta x \]

\[ F_{gx} = mg \sin \theta \]

\[ \Delta x = \frac{h}{\sin \theta} \]

\[ W = mgh \]
Which of the following statements is true?

A) \( W_{Fg}^{(a)} > W_{Fg}^{(b)} \)

B) \( W_{Fg}^{(a)} < W_{Fg}^{(b)} \)

C) \( W_{Fg}^{(a)} = W_{Fg}^{(b)} \)

D) none of the above.
Answer

Which of the following statements is true?

- A) $W_{Fg}^{(a)} > W_{Fg}^{(b)}$
- B) $W_{Fg}^{(a)} < W_{Fg}^{(b)}$
- C) $W_{Fg}^{(a)} = W_{Fg}^{(b)}$
- D) none of the above.
\[ W_g = \int \vec{F}_g \cdot d\vec{s} \]

\[ \vec{F}_g \cdot d\vec{s} = mg ds_y \]

\[ W_g = \int_0^h mg ds_y \]

\[ = mg \int_0^h ds_y \]

\[ = mgh. \]

\[ W = mgh \]
• **Conservative Forces**

The work done by the force depends only on the initial and final position of the object, not on the path in between.

\[\uparrow\]

The net work done by a conservative force on a particle moving around any closed path is zero.
**Consequence:** when the configuration change is reversed the work changes sign:

$$W_{a\rightarrow b} = -W_{b\rightarrow a}$$
• **Examples:** gravitational force, elastic force

• **Potential energy for conservative forces:** define $U$ such that:

$$\Delta U = U_f - U_i = -W_{i\rightarrow f}$$

**Note:**

• $W_{i\rightarrow f}$ is path independent, hence this is a consistent relation

• Choosing $U_0 = 0$ for some reference configuration:

$$U_a = -W_{0\rightarrow a}$$
• Gravitational potential energy

\[ \Delta U_g = mg(y_f - y_i) \]

Reference configuration: ground level

\[ U_g(0) = 0 \Rightarrow U = mgy \]

• Elastic potential energy

\[ \Delta U_s = \frac{k}{2}(x_f^2 - x_i^2) \]

Reference configuration: relaxed spring

\[ U_s(0) = 0 \Rightarrow U_s = \frac{kx^2}{2} \]
Conservation of Mechanical Energy

In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy $E_{mec}$ of the system, cannot change.

Conservative forces, isolated system $\Rightarrow U + K = \text{constant}$
i-Clicker

A block of mass $m$ slides down a curved slope as shown below. What is the final speed of the block?

A) $v = \sqrt{2gy_1}$
B) $v = \sqrt{2gy_2}$
C) $v = \sqrt{2g(y_1 - y_2)}$
D) none of the above
A block of mass $m$ slides down a frictionless curved slope as shown below. What is the final speed of the block?

A) $v = \sqrt{2gy_1}$

B) $v = \sqrt{2gy_2}$

C) $v = \sqrt{2g(y_1 - y_2)}$

D) none of the above

Energy conservation:

$$mg y_1 = mg y_2 + mv^2/2 \implies v = \sqrt{2g(y_1 - y_2)}$$
• **Non-conservative (dissipative) forces:**
  
  • $W$ depends on the path
  
  • There is **no** potential energy $U$ associated to a configuration such that
    
    $$\Delta U = -W$$
  
  • Examples: kinetic friction, drag
Example:

- Suppose an object is launched from $A$ to $B$ on a rough horizontal surface with kinetic friction coefficient $\mu_k$.

1. along a straight line
2. on a circular trajectory (tied to a string)

$W_{A\rightarrow B}^{(1)} = W_{B\rightarrow A}^{(2)}$?
\[ W_{A\rightarrow B}^{(1)} = -\mu_k m g d_{AB} \]

\[ W_{A\rightarrow B}^{(2)} = \int_A^B \vec{f}_k \cdot d\vec{s} = -\frac{\pi}{2} \mu_k m g d_{AB} \]

In conclusion:

\[ W_{A\rightarrow B}^{(1)} \neq W_{B\rightarrow A}^{(2)} \]
A boy is initially seated on the top of a hemispherical ice mound of radius $R = 13.8$ m. He begins to slide down the ice, with a negligible initial speed (Fig. 8-45). Approximate the ice as being frictionless. At what height does the boy lose contact with the ice?
1. Energy conservation:

\[ mgR = mgh + \frac{mv^2}{2} \]

\[ h = \text{height above the ground when losing contact} \]

\[ h = R \cos(\theta) \]

\[ v^2 = 2g(R - h) \]
2. Newton’s law:

\[ N + F_{gy} = -ma_c \]

\[ N = mg \cos(\theta) - \frac{mv^2}{R} \]

\[ N = \frac{mgh}{R} - \frac{2mg(R - h)}{R} \]

\[ N = mg \frac{3h - 2R}{R} \]

**Note:** the \( y \) direction is the radial direction.
Contact is lost when:

\[ N = mg \frac{3h - 2R}{R} \]

\[ h = \frac{2R}{3} \]
Problem 22, Ch. 8: A skier starts from rest at height $H$ above the end of a ramp and leaves the ramp at angle $\theta$. No air resistance, no friction. What is the maximum height $h$ of his jump above the end of the ramp?
What is the total kinetic energy at the end of the ramp?

A) \( mgh \)  
B) 0  
C) \( mgH \)  
D) \( mg(H-h) \)
Is all the kinetic energy at the end of the ramp converted back into gravitational potential energy?

A) Yes  
B) No
At the end of ramp:

\[ v = \sqrt{2gH} \quad v_x = v\cos(\theta) \quad v_y = v\sin(\theta) \]

\[ K = \frac{mv^2}{2} = mgH\cos^2(\theta) + mgH\sin^2(\theta) \]
At the top of jump:

\[ v_x = v \cos(\theta) \quad v_y = 0 \]

\[ K = \frac{mv_x^2}{2} = mgH\cos^2(\theta) \]
Is **all** the kinetic energy at the end of the ramp converted back into gravitational potential energy?

A) Yes

B) **No**  \( K_{\text{top}} = mgH\cos^2(\theta) \)
Energy conservation during jump:

\[ mgH = mgh + mgH\cos^2(\theta) \]

\[ h = H(1 - \cos^2(\theta)) = H\sin^2(\theta) \]
Problems 29, 35, Ch. 8 The distance between the spring and the upper end of the relaxed spring, measured along the ramp, is $d$. No friction. What is the maximum deformation $x$ of the spring relative to its relaxed configuration?
What is the gravitational potential energy relative to the turning point?

A) $mgd$
B) $mgdsin(\theta)$
C) $mg(d + x)\sin(\theta)$
D) 0
What is the gravitational potential energy relative to the turning point?

A) $mgd$

B) $mgdsin(\theta)$

C) $mg(d + x)sin(\theta)$

D) 0
Energy conservation:

\[ mg(d + x)\sin(\theta) = \frac{kx^2}{2} \]
Second degree equation for $x$

$$\frac{kx^2}{2} - mgx\sin(\theta) - mgd = 0$$
What is the kinetic energy of the block when it hits the spring?

A) $mgd$

B) $mgdsin(\theta)$

C) $mg(d + x)$

D) $mg(d + x)sin(\theta)$
What is the kinetic energy of the block when it hits the spring?

A) $mgd$

B) $mgdsin(\theta)$

C) $mg(d + x)$

D) $mg(d + x)sin(\theta)$