Dynamics of uniform circular motion

Centripetal acceleration:

\[ \vec{a} = -\frac{v^2}{r} \hat{r} \]

- \( r \) = radius of the circle
- \( v \) = speed
- \( \hat{r} = \frac{\vec{r}}{r} \) \textbf{unit} radial vector
Newton’s 2nd law:

\[ \vec{F}_{\text{net}} = m\vec{a} \]

\[ \downarrow \]

\[ \vec{F}_{\text{net}} = -\frac{mv^2}{r}\hat{r} \]

**Centripetal Force**

A centripetal force accelerates a body by changing the direction of the body’s velocity without changing the body’s speed.
**Example:** A ball attached to a string of length $l$, which makes an angle $\theta$ with the vertical, rotates uniformly in a horizontal plane as shown below. Find the speed $v$. 

![Diagram of a ball on a string with forces diagram](attachment:image.png)
\[ \vec{F}_{\text{net}} = m\vec{a} \quad \vec{F}_{\text{net}} = \vec{F}_g + \vec{T} \]

\[ (F_{\text{net}})_x = T\sin \theta \]
\[ (F_{\text{net}})_y = -mg + T\cos \theta \]

\[ a_x = \frac{v^2}{r} \quad a_y = 0 \]

\[ T\sin \theta = \frac{mv^2}{r} = \frac{mv^2}{l \sin \theta} \]
\[ T\cos \theta = mg \]

\[ \frac{v^2}{l \sin \theta} = \tan \theta \quad \Rightarrow \quad v = \sin \theta \sqrt{\frac{lg}{\cos \theta}} \]
Example

A ball attached to a string rotates in a vertical plane near Earth’s surface such that the string is stretched taut all points on the circular path. What is the minimum value of its speed at the highest point \( A \) of the trajectory.

\[ A) \ v_{\text{min}} = 0 \]
\[ B) \ v_{\text{min}} = \sqrt{rg} \]
\[ C) \ v_{\text{min}} = \sqrt{rg/2} \]
\[ D) \ v_{\text{min}} = \sqrt{2gr} \]
\[ E) \ \text{none of the above} \]
\[ \vec{F}_{\text{net}} = -\frac{mv^2}{r} \]  
\[ \vec{F}_{\text{net}} = \vec{F}_g + \vec{T} \]

\[(F_{\text{net}})_y = -(mg + T) = -\frac{mv^2}{r} \]

\[ T = \frac{mv^2}{r} - mg \geq 0 \]

\[ v^2 \geq rg \implies v_{\text{min}} = \sqrt{rg} \]
i-Clicker

A ball attached to a string rotates in a vertical plane near Earth’s surface such that the string is stretched taut all points on the circular path.

Is there any horizontal acceleration at the top or at the bottom?

A) Yes, at the top
B) Yes, at the bottom
C) No on both counts
i-Clicker

A ball attached to a string rotates in a vertical plane near Earth's surface such that the string is stretched taut all points on the circular path.

Is there any horizontal acceleration at the top or at the bottom?

A) Yes, at the top
B) Yes, at the bottom
C) No on both counts
\[(F_{\text{net}})_{\text{horizontal}} = 0\]
How about some point in between?

Is the motion \textit{uniform} (constant speed)?
Does the net force have a **tangential** component?
Radial:
\[(F_{net})_y = -T - mg\sin(\theta)\]

Tangential:
\[(F_{net})_x = mg\cos(\theta)\]

Newton’s 2nd law

\[T + mg\sin(\theta) = ma_c\]
\[mg\cos(\theta) = ma_t\]
\[\theta \neq 90^\circ \ \Rightarrow \ a_t \neq 0\]
General circular motion

\[ \vec{a} = \vec{a}_c + \vec{a}_t \]

\[ \vec{a}_c = -\frac{v^2}{r} \hat{r} \]

**Note:** \( v \) and \( a \) may depend on angular displacement and time.
7. Kinetic energy and work

- **What is energy?**

  - **Basic Idea:** scalar quantity with quantifies the state of motion and/or the capacity for motion of a system
  
  - **Conserved** as the state of motion of the system changes (under the right conditions)
Speed of train at the lowest point?

Speed of arrow in flight?
Energy

- **Kinetic energy**: associated to motion
- **Potential energy**: associated to the capacity of generating motion

**Example**: freely falling ball from height $h$.

- **initial speed**: $v_0 = 0$
- **final speed**:

$$v^2 = v_0^2 + 2gh = 2gh$$
• initially: potential energy: $mgh$

• finally: kinetic energy: $\frac{mv^2}{2}$

• Energy conservation:

\[ mgh = \frac{mv^2}{2} \]
• **Kinetic Energy:** energy associated to the motion of an object

\[ K = \frac{1}{2}mv^2 \]

**Unit:** Joule

\[ 1 \text{J} = 1 \text{kg} \times \text{m}^2/\text{s}^2 \]
Non-zero force

⇓

Acceleration: change in velocity

⇓

Change in kinetic energy

How does $K$ change under an applied force?
• **Work**: energy transferred to or from an object by means of a force acting on the object.

Energy transferred to the object is **positive work** while energy transferred from the object is **negative work**.
How do we calculate the work of a force?

- **Example:** bead on frictionless rod subject to constant force $\vec{F}$

Newton’s 2nd law:

$$F_x = ma_x$$

Constant acceleration model:

$$v_x^2 = v_{0x}^2 + 2a_x \Delta x$$

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = F_x \Delta x$$
Work done by the force $\vec{F}$

$$F \cos \phi = \vec{F} \cdot \vec{d}$$

$$\vec{d} = (\Delta x)\hat{i}$$

displacement vector

$$F \cos \phi \begin{cases} 
> 0, & \text{for } 0 \leq \phi < \pi/2 \quad \text{positive work, } K_f > K_i \\
= 0, & \text{for } \phi = \pi/2 \quad \text{zero work, } K_f = K_i \\
< 0, & \text{for } \pi/2 < \phi \leq \pi \quad \text{negative work, } K_f < K_i
\end{cases}$$
• **Net work:**

When two or more forces act on an object, the net work done on the object is the sum of the works done by the individual forces.

\[
W_{net} = \sum W = \sum \vec{F} \cdot \vec{d} = (\sum \vec{F}) \cdot \vec{d} = \vec{F}_{net} \cdot \vec{d}
\]
• Work-Kinetic Energy Theorem

\[
\Delta K = W \quad K_f = K_i + W
\]
An object of mass $m = 1 \text{ kg}$ is launched with initial speed $v_0 = 2 \text{ m/s}$ along a rough horizontal surface. What is the total work done by the frictional force until it stops?

\[ A) \ 2 \text{ J} \]
\[ B) \ 1 \text{ J} \]
\[ C) \ -1 \text{ J} \]
\[ D) \ -2 \text{ J} \]
\[ E) \ 0 \text{ J}. \]
Answer

An object of mass $m = 1 \text{ kg}$ is launched with initial speed $v_0 = 2 \text{ m/s}$ along a rough horizontal surface. What is the total work done by the frictional force until it stops?

\[ W = \Delta K = K_f - K_i = -\frac{mv_0^2}{2} \]

A) 2 J  
B) 1 J  
C) −1 J  
D) −2 J  
E) 0 J.
What if the trajectory is curved?

\[ d\mathbf{s} = \mathbf{v} \, dt \]
\[
\frac{dK}{dt} = \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) = m \frac{d}{dt} \left( \vec{v} \cdot \vec{v} \right) = m \left( \vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} \cdot \vec{v} \right) = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F}_{\text{net}} \cdot \vec{v} = \vec{F}_{\text{net}} \cdot \frac{d\vec{s}}{dt}
\]
\[ \frac{dK}{dt} = \vec{F}_{\text{net}} \cdot \frac{d\vec{s}}{dt} \]

\[ dK = \vec{F}_{\text{net}} \cdot d\vec{s} \]

\[ \Delta K = \int_{\text{trajectory}} \vec{F} \cdot d\vec{s} \]

\[ \delta W = \vec{F}_{\text{net}} \cdot d\vec{s} \]

\[ d\vec{s} = d\vec{r}. \]
A ball of mass $m$ attached to string is launched with initial speed $v_0$ on a circular trajectory on horizontal plane. The friction force between the ball and the surface has constant magnitude $f_k$. What is the speed of the ball after travelling a distance $s$ along the circle.

$A) \quad v = v_0$

$B) \quad v = \sqrt{v_0^2 - \frac{2f_ks}{m}}$

$C) \quad v = v_0 - \frac{f_ks}{mv_0}$

$D) \quad$ cannot be determined from the data.
Answer

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A) $v = v_0$

B) $v = \sqrt{v_0^2 - \frac{2f_ks}{m}}$

C) $v = v_0 - \frac{f_ks}{mv_0}$

D) cannot be determined from the data.
Infinitesimal displacement
\[ \vec{d}s = \vec{v}dt \]
Note that \( T \perp \vec{d}s \) while \( \vec{f}_k \) makes an angle \( \theta = 180^\circ \) with \( \vec{d}s \).
Infinitesimal work:
\[ dW_{f_k} = \vec{f}_k \cdot \vec{d}s = -f_k ds \]
Total work:
\[ W = \int dW = -f_k s \]
Work-energy theorem:
\[ \frac{mv^2}{2} - \frac{mv_0^2}{2} = -f_k s \]
\[ v = \sqrt{v_0^2 - \frac{2f_k s}{m}} \]
Work done by the gravitational force

Freely falling object moving downwards:

\[ W = \vec{F}_g \cdot \vec{d} = mgd \cos 0^\circ = mgd > 0 \]

\[ K_1 = \frac{mv_1^2}{2} \quad K_2 = \frac{mv_2^2}{2} \]

\[ K_2 - K_1 = mgd > 0 \]
Freely falling object moving upwards:

\[ W = \vec{F}_g \cdot \vec{d} = mgd\cos 180^\circ = -mgd > 0 \]

\[ K_1 = \frac{mv_1^2}{2} \quad K_2 = \frac{mv_2^2}{2} \]

\[ K_2 - K_1 = -mgd < 0 \]
An object of mass $m$ is launched with initial speed $v_0$ along an inclined plane making an angle $\theta = 45^\circ$ with the horizontal. The kinetic friction coefficient between the object and the plane is $\mu_k = 0.5$. Let $W_{fk}$ be the total work done by the friction force until it stops. Which of the following statements is false?

A) $W_{fk} < 0$

B) $W_{fk} = -mv_0^2/2$

C) $|W_{fk}| < mv_0^2/2$
Answer

An object of mass $m$ is launched with initial speed $v_0$ along an inclined plane making an angle $\theta = 45^\circ$ with the horizontal. The kinetic friction coefficient between the object and the plane is $\mu_k = 0.5$. Let $W_{fk}$ be the total work done by the friction force until it stops. Which of the following statements is false?

$A) \ W_{fk} < 0$

$B) \ W_{fk} = -\frac{mv_0^2}{2}$

$C) \ |W_{fk}| < \frac{mv_0^2}{2}$
\[ F_{\text{net}} = m\vec{a} \]
\[ (F_{\text{net}})_x = ma_x \quad (F_{\text{net}})_y = ma_y = 0 \]

\[ F_{\text{net}} = \vec{F}_g + \vec{F}_N + \vec{f}_k \]

\[ (F_{\text{net}})_x = -mg\sin\theta - f_k \quad (F_{\text{net}})_y = F_N - mg\cos\theta = 0 \]

\[ F_N = mg\cos\theta \quad f_k = \mu_k mg\cos\theta \]
Work done by kinetic friction:

\[ W_{f_k} = \vec{f}_k \cdot \vec{d} = -f_k d = -\mu_k mg d \cos \theta \]

Work done by gravitational force:

\[ W_{F_g} = \vec{F}_g \cdot \vec{d} = -mgh = -mg d \sin \theta \]

Work done by normal force

\[ \vec{F}_N \cdot \vec{d} = 0 \]
Total work:
\[ W = W_{fk} + W_{Fg} = -mgd (\sin \theta + \mu_k \cos \theta) \]

Work-Kinetic energy theorem:
\[ W = \Delta K = 0 - K_i = -\frac{mv_0^2}{2} \]

Work done by kinetic friction:
\[
\begin{align*}
\frac{W_{fk}}{|W|} &= -\frac{\mu_k \cos \theta}{\sin \theta + \mu_k \cos \theta} = \frac{1}{3} \\
W_{fk} &= -\frac{mv_0^2}{6}
\end{align*}
\]