Midterm I

15 14 13 12 11 10 9 8 7 6 5
6 19 15 16 10 12 12 7 5 9 2

Class average

$$\frac{10.86}{15} = 72.43\%$$
• Properties of friction

1. For a stationary object \( \vec{F} + \vec{f}_s = 0 \) i.e. the applied force and the static friction balance each other out.

2. The magnitude of static friction has a maximum value depending on the magnitude of the normal force \( F_N \):

\[
(f_s)_{\text{max}} = \mu_s F_N
\]

where \( \mu_s \) = coefficient of static friction. When \( F \geq (f_s)_{\text{max}} \) the object begins to slide.

3. When sliding begins, the magnitude of friction rapidly decreases to a value

\[
f_k = \mu_k F_N, \quad \mu_k = \text{coefficient of kinetic friction}
\]
• Example:

A horizontal tension force \( \vec{T} \) is applied to an object of mass \( m \) on an inclined plane. The object moves upwards along the plane with constant acceleration \( \vec{a} \).

Find the kinetic friction coefficient \( \mu_k \) between the object and the plane.

For which values of \( T, a \) is this motion possible?
The $x$ axis is $\parallel$ to the ramp and points uphill. What is the $x$-component of the net force?

A) $-mg \sin(\theta)$

B) $T \cos(\theta)$

C) $-mg \sin(\theta) - f_k$

D) $T \cos(\theta) - mg \sin(\theta) - f_k$

E) None of the above.
The $x$ axis is $\parallel$ to the ramp and points uphill. What is the $x$-component of the net force?

A) $-mg\sin(\theta)$

B) $T\cos(\theta)$

C) $-mg\sin(\theta) - f_k$

D) $T\cos(\theta) - mg\sin(\theta) - f_k$

E) None of the above.
\[(F_{\text{net}})_x = T\cos\theta - mg\sin\theta - f_k\]
The $y$ axis is $\perp$ to the ramp and points upward. What is the $y$-component of the net force?

A) $-mg\cos(\theta)$

B) $T\sin(\theta)$

C) $-mg\cos(\theta) + F_N$

D) $-T\sin(\theta) - mg\cos(\theta) + F_N$

E) None of the above.
i-Clicker

A) \(-mg\cos(\theta)\)
B) \(T\sin(\theta)\)
C) \(-mg\cos(\theta) + F_N\)
D) \(-T\sin(\theta) - mg\cos(\theta) + F_N\)
E) None of the above.
\[(F_{\text{net}})_y = -T\sin\theta - mg\cos\theta + F_N\]
Newton’s 2nd law:

\[ \vec{F}_{\text{net}} = m\vec{a} \]

\[ (F_{\text{net}})_x = ma_x \quad (F_{\text{net}})_y = 0 \]

\[ (F_{\text{net}})_x = T\cos\theta - mg\sin\theta - f_k \]

\[ (F_{\text{net}})_y = -T\sin\theta - mg\cos\theta + F_N \]

\[ a_x = |\vec{a}| = a \quad (\text{moving uphill}) \]

\[ f_k = T\cos\theta - mg\sin\theta - ma, \quad f_k = \mu_k F_N \]

\[ F_N = T\sin\theta + mg\cos\theta \]

\[ \mu_k = \frac{T\cos\theta - mg\sin\theta - ma}{T\sin\theta + mg\cos\theta} \]
For which values of $T, a$ is the motion possible?

\[
\mu_k = \frac{T\cos\theta - mg\sin\theta - ma}{T\sin\theta + mg\cos\theta} \geq 0
\]

\[
T\cos\theta \geq m(g\sin\theta + a)
\]
• Drag force and terminal speed

Drag force:
• force caused by relative motion of an object and a fluid
• opposed the relative motion and points in the direction the fluid flows relative to the object
For a fast blunt object moving through air such that the flow becomes turbulent

\[ D = \frac{1}{2} C \rho A v^2 \]

- \( A = \text{effective crossection}: \text{area of crossection } \perp \vec{v} \)
- \( C = \text{drag coefficient} \)
- \( \rho = \text{air density} \)
Terminal velocity:

Body falling from rest through air:

The drag force increases gradually until it balances the gravitational force.

The body reaches terminal velocity.

\[
\vec{D} + \vec{F}_g = 0
\]

\[
\frac{1}{2} C A \rho v_y^2 - mg = 0 \quad \Rightarrow \quad v_y = -\sqrt{\frac{2mg}{C \rho A}}
\]
A sky diver jumps from a flying airplane and falls for several seconds before she reaches terminal velocity. She then opens her parachute, reaches a new terminal velocity, and continues her descent to the ground. Which one of the following graphs of the drag force versus time best represents this situation?
A sky diver jumps from a flying airplane and falls for several seconds before she reaches terminal velocity. She then opens her parachute, reaches a new terminal velocity, and continues her descent to the ground. Which one of the following graphs of the drag force versus time best represents this situation?
Before opening the parachute terminal velocity is reached when

\[ \vec{D} + \vec{F}_g = 0 \quad D = F_g \]

After opening the parachute terminal velocity is again reached when

\[ \vec{D} + \vec{F}_g = 0 \quad D = F_g \]

A sufficiently long time after opening the magnitude of the drag force must equal its value before opening. At the same time it should spike upwards during opening.
Theoretically, how long does it take to reach terminal velocity?

(A) $\Delta t = \sqrt{\frac{2m}{gC\rho A}}$

(B) $\Delta t = 0$

(C) Infinite amount of time.

(D) None of the above.
Answer

Theoretically, how long does it take to reach terminal velocity?

(A) $\Delta t = \sqrt{\frac{2m}{gC\rho A}}$

(B) $\Delta t = 0$

(C) Infinite amount of time.

(D) None of the above.

Why?
\[ \vec{F}_{\text{net}} = m\vec{a} \quad \vec{F}_{\text{net}} = \vec{D} + \vec{F}_g \]

\[ (F_{\text{net}})_y = \frac{1}{2}CA\rho v^2 - mg \]

\[ \frac{dv_y}{dt} = \kappa v_y - mg, \quad \kappa = \frac{1}{2}CA\rho \]

\[ v_y = -\sqrt{\frac{mg}{\kappa}} \tanh(\sqrt{mg\kappa t}) = -\sqrt{\frac{mg}{\kappa}} \frac{e^{\sqrt{mg\kappa t}} - e^{-\sqrt{mg\kappa t}}}{e^{\sqrt{mg\kappa t}} + e^{-\sqrt{mg\kappa t}}} \]
Terminal velocity:

\[
\lim_{t \to \infty} v_y = -\sqrt{\frac{mg}{\kappa}} = -\sqrt{\frac{2mg}{C\rho A}}
\]

\[
\lim_{t \to \infty} a_y = \lim_{t \to \infty} \frac{dv_y}{dt} = 0
\]
<table>
<thead>
<tr>
<th>Object</th>
<th>Terminal Speed (m/s)</th>
<th>95% Distance (^a) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shot (from shot put)</td>
<td>145</td>
<td>2500</td>
</tr>
<tr>
<td>Sky diver (typical)</td>
<td>60</td>
<td>430</td>
</tr>
<tr>
<td>Baseball</td>
<td>42</td>
<td>210</td>
</tr>
<tr>
<td>Tennis ball</td>
<td>31</td>
<td>115</td>
</tr>
<tr>
<td>Basketball</td>
<td>20</td>
<td>47</td>
</tr>
<tr>
<td>Ping-Pong ball</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>Raindrop (radius = 1.5 mm)</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Parachutist (typical)</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

\(^a\)This is the distance through which the body must fall from rest to reach 95\% of its terminal speed.

Source: Adapted from Peter J. Brancazio, Sport Science, 1984, Simon & Schuster, New York.
4. Uniform circular motion in 2D

- Particle moving around a circle or circular arc
- The speed $|\vec{v}|$ is constant.

Is the acceleration vector zero?
\[ \vec{a} = \frac{d\vec{v}}{dt} \quad \vec{a} = 0 \iff \vec{v} \text{ has constant magnitude and constant direction} \]

Average acceleration:

\[ \vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t} \neq 0 \]
i-Clicker

What is the angle between $\Delta \vec{r}$ and $\vec{a}_{avg}$, measured counterclockwise?

A) $45^\circ$

B) $60^\circ$

C) $90^\circ$

D) $120^\circ$

E) None of the above.
Answer
What is the angle between $\Delta \vec{r}$ and $\vec{a}_{avg}$, measured counterclockwise?

A) $45^\circ$

B) $60^\circ$

C) $90^\circ$

D) $120^\circ$

E) None of the above.
\[ \Delta \vec{v} = \vec{v}_2 - \vec{v}_1 = v(-\hat{i} - \hat{j}) \]
\[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 = R(\hat{j} - \hat{i}) \]
\[ \Delta \vec{r} \cdot \Delta \vec{v} = 0 \]
i-Clicker
What is the angle between $\Delta \vec{r}$ and $\vec{a}_{avg}$, measured counterclockwise, for some arbitrary displacement?

A) Still $90^\circ$

B) Depends on displacement
Answer
What is the angle between $\Delta \vec{r}$ and $\vec{a}_{avg}$, measured counterclockwise, for some arbitrary displacement?

A) Still 90°

B) Depends on displacement
\[ \vec{r} = x\hat{i} + y\hat{j} \]
\[ \vec{v} = v_x\hat{i} + v_y\hat{j} \]
\[ \vec{r}^2 = x^2 + y^2 = R^2 \]
\[ \vec{v}^2 = v_x^2 + v_y^2 = v^2 \]
\[ \vec{v} \cdot \vec{r} = xv_x + yv_y = 0 \]
\[ \Rightarrow \]
\[ \vec{v} = \frac{v}{R} (-y\hat{i} + x\hat{j}) \]
\[
\vec{v}_1 = \frac{v}{R} \left( -y_1 \hat{i} + x_1 \hat{j} \right)
\]
\[
\vec{v}_2 = \frac{v}{R} \left( -y_2 \hat{i} + x_2 \hat{j} \right)
\]
\[
\Delta \vec{v} = \frac{v}{R} \left( -\Delta y \hat{i} + \Delta x \hat{j} \right)
\]
\[
(\vec{v}_2 - \vec{v}_1) \cdot (\vec{r}_2 - \vec{r}_1) = \frac{v}{R} \left( -\Delta y \hat{i} + \Delta x \hat{j} \right) \cdot \left( \Delta x \hat{i} + \Delta y \hat{j} \right) = \frac{v}{R} ( -\Delta x \Delta y + \Delta x \Delta y ) = 0
\]
\[
\vec{a}_{\text{avg}} \cdot \vec{v}_{\text{avg}} = 0 \quad \text{for any displacement}
\]
$\vec{a}$ is parallel to $\vec{r}$ and points into opposite direction.

\[ \Delta t \to 0 \]

**Instantaneous acceleration:**

\[ \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \neq 0 \]
• **Centripetal (center-seeking) Acceleration**

The acceleration vector always points toward the center.

\[ \vec{a} = -\frac{v^2}{r} \hat{r} \]

\[ |\vec{a}| = \frac{v^2}{r} \]

\[ r = \text{radius of the circle} \]

\[ v = \text{speed} \]

\[ \hat{r} = \frac{\vec{r}}{r} \text{ unit radial vector} \]
Proof:

\[ v_x = -v \sin \theta = -v \frac{y_p}{r} \]
\[ a_x = \frac{dv_x}{dt} = -\frac{v}{r} \frac{dy_p}{dt} \]
\[ v_y = v \cos \theta = v \frac{x_p}{r} \]
\[ a_y = \frac{dv_y}{dt} = \frac{v}{r} \frac{dx_p}{dt} \]
\[
\frac{dy_p}{dt} = v_y = v \cos \theta = \frac{v x_p}{r}
\]

\[
\frac{dx_p}{dt} = v_x = -v \sin \theta = -\frac{v y_p}{r}
\]

\[
a_x = -\frac{v}{r} \frac{dy_p}{dt} = -\frac{v^2 x_p}{r^2} \quad a_y = \frac{v}{r} \frac{dx_p}{dt} = -\frac{v^2 y_p}{r^2}
\]
\[ a_x = -\frac{v \, dy_p}{r \, dt} = -\frac{v^2 x_p}{r \, r} \quad a_y = \frac{v \, dx_p}{r \, dt} = -\frac{v^2 y_p}{r \, r} \]

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} = -\frac{v^2}{r} \left( \frac{x_p \hat{i} + y_p \hat{j}}{r} \right) = -\frac{v^2 \vec{r}}{r \, r} = -\frac{v^2 \hat{r}}{r} \]

\[ \vec{a} = -\frac{v^2}{r} \hat{r} \]

\[ |\vec{a}| = \frac{v^2}{r} \]
• **Period**

\[ T = \frac{2\pi r}{v} \]

how long for one rotation

• **Frequency**

\[ \nu = \frac{1}{T} = \frac{v}{2\pi r} \]

number of rotations per unit time
• Dynamics of uniform circular motion

Centripetal acceleration:

\[ \vec{a} = -\frac{v^2}{r} \hat{r} \]

\[ r = \text{radius of the circle} \]
\[ v = \text{speed} \]
\[ \hat{r} = \frac{\vec{r}}{r} \quad \text{unit radial vector} \]
Newton’s 2nd law:

\[ \vec{F}_{\text{net}} = m\vec{a} \]

\[ \downarrow \]

\[ \vec{F}_{\text{net}} = -\frac{mv^2}{r} \hat{r} \]

Centripetal Force

A centripetal force accelerates a body by changing the direction of the body’s velocity without changing the body’s speed.
• ball attached to a string in uniform circular motion on a horizontal frictionless plane

\[ \vec{F}_{\text{net}} = \vec{T} \text{ (tension)} \]

• the string breaks ⇒ uniform linear motion, according to Newton’s 1st law.

• no \( \vec{T} \) ⇒ no centripetal force ⇒ \( \vec{a} = 0 \)
i-Clicker
A ball attached to a string rotates in a vertical plane near Earth's surface such that the string is stretched taut all points on the circular path. What is the minimum value of its speed at the highest point $A$ of the trajectory.

\[
\begin{align*}
A) \quad v_{\text{min}} &= 0 \\
B) \quad v_{\text{min}} &= \sqrt{rg} \\
C) \quad v_{\text{min}} &= \sqrt{rg/2} \\
D) \quad v_{\text{min}} &= \sqrt{2gr} \\
E) \quad \text{none of the above}
\end{align*}
\]
A ball attached to a string rotates in a vertical plane near Earth’s surface. For a point $P$ on the trajectory, let $T_P$ denote the magnitude of the tension in the string. Which of the following statements is true?

A) $v_{\text{min}} = 0$

B) $v_{\text{min}} = \sqrt{rg}$

C) $v_{\text{min}} = \sqrt{rg/2}$

D) $v_{\text{min}} = \sqrt{2gr}$

E) none of the above
\[
\vec{F}_{\text{net}} = -\frac{mv^2}{r} \hat{r} \quad \vec{F}_{\text{net}} = \vec{F}_g + \vec{T}
\]

\[
(F_{\text{net}})_y = -(mg + T) = -\frac{mv^2}{r}
\]

\[
T = \frac{mv^2}{r} - mg \geq 0
\]

\[
v^2 \geq rg \implies v_{\min} = \sqrt{rg}
\]
What is the tension $T$ at the lowest point?

\[ m\vec{a}_c = \vec{T} + \vec{F}_g \]

\[ \frac{mv^2}{R} = T - mg \]

\[ T = mg + \frac{mv^2}{R} \]
i-Clicker

A ball attached to a string rotates in a vertical plane near Earth’s surface such that the string is stretched taut all points on the circular path.

Is there any horizontal acceleration at the top or at the bottom?

A) Yes, at the top
B) Yes, at the bottom
C) No on both counts
A ball attached to a string rotates in a vertical plane near Earth’s surface such that the string is stretched taut all points on the circular path.

Is there any horizontal acceleration at the top or at the bottom?

A) Yes, at the top
B) Yes, at the bottom
C) No on both counts
(F_{\text{net}})_{\text{horizontal}} = 0
How about some point in between?

Is the motion uniform (constant speed)?
Does the net force have a tangential component?
Radial:
\[(F_{\text{net}})_y = -T - mg\sin(\theta)\]

Tangential:
\[(F_{\text{net}})_x = mg\cos(\theta)\]

Newton’s 2nd law

\[T + mg\sin(\theta) = ma_c\]
\[mg\cos(\theta) = ma_t\]

\[\theta \neq 90^\circ \Rightarrow a_t \neq 0\]
**General** circular motion

\[ \vec{a} = \vec{a}_c + \vec{a}_t \]

\[ \vec{a}_c = -\frac{v^2}{r} \hat{r} \]

**Note:** \( v \) and \( a \) may depend on angular displacement and time
• **Example:** A ball attached to a string of length $l$, which makes an angle $\theta$ with the vertical, rotates uniformly in a horizontal plane as shown below. Find the speed $v$. 

![Diagram showing a ball attached to a string with forces $T_y$, $T_x$, and $F_g$.]
\[ \vec{F}_{\text{net}} = m\vec{a} \quad \vec{F}_{\text{net}} = \vec{F}_g + \vec{T} \]

\[
(F_{\text{net}})_x = T\sin \theta \\
(F_{\text{net}})_y = -mg + T\cos \theta
\]

\[
a_x = \frac{v^2}{r} \quad a_y = 0
\]

\[
T\sin \theta = \frac{mv^2}{r} = \frac{mv^2}{l\sin \theta} \\
T\cos \theta = mg
\]

\[
\frac{v^2}{l\sin \theta} = \tan \theta \quad \Rightarrow \quad v = \sin \theta \sqrt{\frac{lg}{\cos \theta}}
\]
• **i-Clicker:** A car of mass $m$ moving with constant speed makes a turn of radius $r$. Suppose the static friction coefficient between the wheels and the road is $\mu_s$. Find the maximum value of the speed $v$ such that the car will remain on the road.

A) Speed does not matter

B) $v_{\text{max}} = \sqrt{gr}$

C) $v_{\text{max}} = \sqrt{\mu_s gr}$

D) $v_{\text{max}} = \sqrt{2\mu_s gr}$

E) $v_{\text{max}} = \sqrt{gr/\mu_s}$
**i-Clicker:** A car of mass $m$ moving with constant speed makes a turn of radius $r$. Suppose the static friction coefficient between the wheels and the road is $\mu_s$. Find the maximum value of the speed $v$ such that the car will remain on the road.

A) Speed does not matter  
B) $v_{max} = \sqrt{gr}$  
C) $v_{max} = \sqrt{\mu_s gr}$  
D) $v_{max} = \sqrt{2\mu_s gr}$  
E) $v_{max} = \sqrt{gr/\mu_s}$
\[
\vec{F}_{\text{net}} = m\vec{a} \\
\vec{a} = -\frac{mv^2}{r} \hat{i}
\]

\[
F_N - mg = ma_y = 0
\]

\[
-f_s = ma_x = -\frac{mv^2}{r}
\]

\[
f_s \leq \mu_s F_N \quad \Rightarrow \quad \frac{mv^2}{r} \leq \mu_s mg
\]

\[
v \leq \sqrt{\mu_s rg}
\]