Midterm 1

• Sunday October 14th 12:00-1:20 pm
• Physics Lecture Hall
• Lectures 1-4 (one dim motion – 2d and 3d motion, relative motion)
• No forces!
4. Motion in two and three dimensions

Position and Displacement

• **3D position vector**
  \[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]
  starting at the origin \( O \).

• \( x, y, z \) are the **components** of \( \vec{r} \), also called the **coordinates** of the particle.

• The path of the particle is generally a **curve**.
• **Displacement:** the change of the position vector \( \vec{r} \) over a time interval \( \Delta t \).

\[
\Delta \vec{r} = (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})
\]
\[
= (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}.
\]
\[
= \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k}.
\]

\[
\vec{r}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} \quad \text{at time } t_1
\]
\[
\vec{r}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \quad \text{at time } t_2
\]
Average and Instantaneous Velocity

- **Average velocity** over a time interval $\Delta t$

\[
\vec{v}_{avg} = \frac{\text{Displacement vector}}{\text{time interval}} = \frac{\Delta \vec{r}}{\Delta t}
\]
• **Instantaneous velocity** at time $t$

\[
\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \left( \frac{dx}{dt} \right) \hat{i} + \left( \frac{dy}{dt} \right) \hat{j} + \left( \frac{dz}{dt} \right) \hat{k}
\]

**Note:**

• $\vec{v}$ is always **tangent** to the path of the particle at its current position.
• **Components** of $\vec{v}$:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$
Average and Instantaneous Acceleration

- **Average acceleration** over a time interval $\Delta t$

$$\vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t}$$
• **Instantaneous acceleration** at time \( t \)

\[
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \left( \frac{dv_x}{dt} \right) \hat{i} + \left( \frac{dv_y}{dt} \right) \hat{j} + \left( \frac{dv_z}{dt} \right) \hat{k}
\]

• **Acceleration** is a vector which measures the change of the velocity vector.

• **Components** of \( \vec{a} \):

\[
a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}
\]
• The acceleration is non-zero if
  
either
  
  the magnitude of $\vec{v}$ changes.
  
or
  
  the direction of $\vec{v}$ changes.

• The acceleration is 0 if and only if both the magnitude and the direction of $\vec{v}$ are constant.
A particle moves on a path as shown below such that the **magnitude** of its velocity vector is **constant**. Where is the instantaneous acceleration \( \vec{a} \) zero?

- **A)** Everywhere
- **B)** At \( A \)
- **C)** At \( C \)
- **D)** At \( B \)
- **E)** Nowhere
Answer

A particle moves on a path as shown below such that the **magnitude** of its velocity vector is **constant**. Where is the instantaneous acceleration $\vec{a}$ zero?

- **A)** Everywhere
- **B)** At $A$
- **C)** At $C$
- **D)** At $B$
- **E)** Nowhere

$|\vec{v}|$ constant $\Rightarrow$ $\vec{v}$ constant on linear segment; $\vec{v}$ **not** constant on curved segments; changes direction.
A particle moves on a helix as shown below such that the \( y \)-component of its velocity is constant. Which of the following statements is false?

A) \( a_x \neq 0 \)
B) \( a_z \neq 0 \)
C) \( a_y = 0 \)
D) \( a_y \neq 0 \)
E) None of the above.
A particle moves on a helix as shown below such that the $y$-component of its velocity is constant. Which of the following statements is false?

- A) $a_x \neq 0$
- B) $a_z \neq 0$
- C) $a_y = 0$
- D) $a_y \neq 0$
- E) None of the above.
• **Projectile Motion:** motion in a vertical plane with constant acceleration equal to the free fall acceleration.
Examples: tennis, baseball, basketball, football …

How hard and at what angle should the quarterback throw?

How should the gun shoot in order to hit the target?
• Set up coordinate system

Acceleration

Initial velocity

Note: no correlation between $v_{0x}, v_{0y}$; can take any values independently from each other.
Recall that
\[ \vec{a} = \frac{d\vec{v}}{dt} \]

Then what is the time dependence of \( v_x \)?

\[ a_x = 0 \quad a_y = -g \]

\[ \vec{a} = -g \hat{j} \]

\[ v_{0x} \]

\[ v_{0y} \]

A) \( v_x = 0 \)
B) \( v_x = v_{0x} \)
C) \( v_x = v_{0x} - gt \)
D) \( v_x = v_{0x} + gt \)
E) \( v_x = v_{0y} \)
Answer

Recall that

\[ \vec{a} = \frac{d\vec{v}}{dt} \]

Then what is the time dependence of \( v_x \)?

A) \( v_x = 0 \)

B) \( v_x = v_{0x} \)

C) \( v_x = v_{0x} - gt \)

D) \( v_x = v_{0x} + gt \)

E) \( v_x = v_{0y} \)

\[ \frac{dv_x}{dt} = a_x = 0 \Rightarrow v_x = \text{constant} = v_{0x} \]
Recall that
\[ \vec{a} = \frac{d\vec{v}}{dt} \]
Then what is the time dependence of \( v_y \)?

A) \( v_y = 0 \)
B) \( v_y = v_{0y} \)
C) \( v_y = v_{0y} - gt \)
D) \( v_y = v_{0y} + gt \)
E) \( v_y = v_{0x} \)
Recall that
\[
\vec{a} = \frac{d\vec{v}}{dt}
\]
Then what is the time dependence of \( v_y \)?

\[\begin{align*}
A) \quad v_y &= 0 \\
B) \quad v_y &= v_{0y} \\
C) \quad v_y &= v_{0y} - gt \\
D) \quad v_y &= v_{0y} + gt \\
E) \quad v_y &= v_{0x}
\end{align*}\]

\[
\frac{dv_y}{dt} = a_y = -g \quad \Rightarrow \quad v_y = v_{0y} - gt
\]
Is there any correlation between vertical and horizontal motion?

No!

\[ v_x = v_{0x}, \quad v_y = v_{0y} - gt \]

\( v_{0x}, v_{0y} \) are completely independent.

Can decompose projectile motion into vertical and horizontal parts.
i-Clicker

A ball is thrown straight up with velocity vector \( \vec{v}_2 \) from a truck moving with constant velocity \( \vec{v}_1 \). For which values of \( v_1, v_2 \) will the ball land back in the truck at a later time?

A) \( v_1 < v_2 \)

B) \( v_1 > v_2 \)

C) \( v_1 = v_2 \)

D) It will not happen for any values of \( v_1, v_2 \).

E) For any values of \( v_1, v_2 \)
**Answer**

A ball is thrown straight up with velocity vector \( \vec{v}_2 \) from a truck moving with constant velocity \( \vec{v}_1 \). For which values of \( v_1, v_2 \) will the ball land back in the truck at a later time?

A) \( v_1 < v_2 \)

B) \( v_1 > v_2 \)

C) \( v_1 = v_2 \)

D) It will not happen for any values of \( v_1, v_2 \).

E) For any values of \( v_1, v_2 \)

The horizontal component \( v_{0x} \) for the ball is equal to \( v_1 \). Since \( a_x = 0 \), \( v_x = v_1 \) at any time \( t > 0 \).
Projectile motion analysed

Initial velocity

\[ \vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}, \quad v_{0x} = v_0\cos\theta_0, \quad v_{0y} = v_0\sin\theta_0 \]
Horizontal motion

\[ a_x = 0 \]

\[ v_x = v_{0x} \]
\[ = v_0 \cos \theta_0 \]

\[ x - x_0 = v_{0x}t \]
\[ = (v_0 \cos \theta_0) t \]

Vertical motion

\[ a_y = -g \]

\[ v_y = v_{0y} - gt \]
\[ = v_0 \sin \theta_0 - gt \]

\[ y - y_0 = v_{0y}t - gt^2/2 \]
\[ = (v_0 \sin \theta_0) t - gt^2/2 \]

\[ v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0) \]
• **Trajectory** \((\theta_0 \neq \pi/2)\)

\[
x - x_0 = (v_0 \cos \theta_0) t, \quad y - y_0 = (v_0 \sin \theta_0) t - \frac{gt^2}{2}
\]

\[\Downarrow\quad \text{Eliminate } t : \quad t = \frac{x - x_0}{v_0 \cos \theta_0}\]

\[
y = y_0 + (\tan \theta_0)(x - x_0) - \frac{g(x - x_0)^2}{2(v_0 \cos \theta_0)^2}
\]

*Parabola*
• **Horizontal range**

The **horizontal** distance $R$ travelled until returning to initial height.

$$y - y_0 = (\tan \theta_0) (x - x_0) - \frac{g(x - x_0)^2}{2(v_0\cos \theta_0)^2}$$

$$y = y_0 \Rightarrow (\tan \theta_0) R - \frac{gR^2}{2(v_0\cos \theta_0)^2}$$

$$R = \frac{v_0^2 \sin (2\theta_0)}{g} \quad \left(\text{using } 2 \sin \theta_0 \cos \theta_0 = \sin(2\theta_0)\right)$$
• **Maximum horizontal range**

For fixed $v_0$, when is $R$ maximum?

$$0 \leq \sin(2\theta_0) \leq 1, \quad \sin(2\theta_0) = 1 \iff 2\theta_0 = \pi/2$$

Hence $R$ is maximum for $\theta_0 = \pi/4$ and

$$R_{max} = \frac{v_0^2}{g}$$
A projectile is launched from the origin as shown below. How long will it take for it to reach maximum height?

\[ A) \frac{v_0}{g} \sin \theta_0 \]
\[ B) \frac{v_0}{g} \cos \theta_0 \]
\[ C) \frac{v_0}{g} \]
\[ D) \frac{v_0}{g} \tan \theta_0 \]
\[ E) \text{None of the above.} \]
Answer

A projectile is launched from the origin as shown below. How long will it take for it to reach maximum height?

\[ A) \frac{v_0}{g} \sin \theta_0 \]
\[ B) \frac{v_0}{g} \cos \theta_0 \]
\[ C) \frac{v_0}{g} \]
\[ D) \frac{v_0}{g} \tan \theta_0 \]
\[ E) \text{None of the above.} \]

\[ v_y = v_0 \sin \theta_0 - gt. \text{ Max height: } v_y = 0 \Rightarrow t = \frac{v_0}{g} \sin \theta_0 \]
i-Clicker

A projectile is launched from the origin as shown below. What is its maximum height?

A) \( \frac{v_0^2}{g} \sin \theta_0 \)
B) \( \frac{v_0^2}{2g} \cos \theta_0 \)
C) \( \frac{v_0^2}{2g} \)
D) \( \frac{(v_0 \sin \theta_0)^2}{2g} \)
E) None of the above.
Answer

A projectile is launched from the origin as shown below. What is its maximum height?

A) \( \frac{v_0^2}{g} \sin \theta_0 \)
B) \( \frac{v_0^2}{2g} \cos \theta_0 \)
C) \( \frac{v_0^2}{2g} \)
D) \( \frac{(v_0 \sin \theta_0)^2}{2g} \)
E) None of the above.

\[ v_y^2 = (v_0 \sin \theta_0)^2 - 2gy; \quad v_y = 0 \Rightarrow y = \frac{(v_0 \sin \theta_0)^2}{2g} \]
Problem 38 Chapter 4

A golf ball is struck at ground level at \( t = 0 \).

(a) How far does the golf ball travel horizontally before returning to ground level?

(b) What is the maximum height above ground level attained by the ball?

Speed graph

\[ v_a = 19 \text{ m/s}, \quad v_b = 31 \text{ m/s} \]
At any time during motion

\[ v = \sqrt{v_x^2 + v_y^2} \]

where

\[ v_x = v_{0x} = \text{constant} \]
\[ v_y = v_{0y} - gt \]

At the highest point on trajectory

\[ v_{0y} = 0 \implies v_{\text{min}} = v_{0x} = v_a \]

At ground level

\[ v_{\text{max}} = \sqrt{v_{0x}^2 + v_{0y}^2} = v_b \]

Hence

\[ v_{0y} = \sqrt{v_b^2 - v_a^2} = 24.49489743 \ldots \text{ m/s} \]
Flying time

\[ \Delta t = \frac{2v_{0y}}{g} \]

Horizontal range

\[ \Delta x = v_{0x}\Delta t = \frac{2v_a}{g} \sqrt{v_b^2 - v_a^2} = 94.98021451 \ldots \text{ m} \]

Maximum height

\[ \frac{v_{0y}^2}{2g} = 2g\Delta y \]

\[ \Delta y = \frac{v_b^2 - v_a^2}{2g} = 30.61224490 \ldots \text{ m}. \]
Problem 41 Chapter 4

An archer fish squirts water drops at an insect on a twig. Although the fish sees the insect along a straight-line path at angle $\phi$ and distance $d$, a drop must be launched at a different angle $\theta$ if its parabolic path is to intersect the insect. Find $\theta$ such that the drop hits the insect at the top of its parabolic path.

\[ d = 0.9 \, m, \quad \phi = 36.0^\circ \]
Maximum height

\[ y_{max} = \frac{v_{0y}^2}{2g} = dsin(\phi) \]

\[ v_{0y} = \sqrt{2gd\sin(\phi)} \]

Flying time until maximum height

\[ \Delta t = \frac{v_{0y}}{g} = \sqrt{\frac{2dsin(\phi)}{g}} \]

Horizontal displacement until maximum height

\[ \Delta x = v_{0x}\Delta t = d\cos(\phi) \]

\[ v_{0x} = \frac{d\cos(\phi)}{\Delta t} = \sqrt{\frac{gd}{2\sin(\phi)\cos(\phi)}} \]
\[ v_{0y} = \sqrt{2gd\sin(\phi)} \]

\[ v_{0x} = \sqrt{\frac{gd}{2\sin(\phi)}} \cos(\phi) \]

\[ \tan(\theta) = \frac{v_{0y}}{v_{0x}} = 2\tan(\phi) \]

\[ \theta = \arctan(2\tan(36^\circ)) = 55.46460289^\circ \ldots \]
Archerfish of the species *Toxotes jaculatrix* take down insects in Indonesia. Photograph by A&J Visage, Alamy
Relative motion in one dimension

- Cars travel at 50-60 mph yet they barely move with respect to each other.
• passengers stand still on moving walkway yet they move with respect to the ground.

The velocity of a particle depends on the reference frame of the observer.
What is the velocity of car $P$ measured by the driver of Car $B$?

$x_{BA}$ coordinate of $B$ in frame $A$

$x_{PA}$ coordinate of $P$ in frame $A$

$x_{PB}$ coordinate of $P$ in frame $B$

$x_{PA} = x_{PB} + x_{BA}$
\( x_{BA} \) coordinate of \( B \) in frame \( A \)

\( x_{PA} \) coordinate of \( P \) in frame \( A \)

\( x_{PB} \) coordinate of \( P \) in frame \( B \)

\( x_{PA} = x_{PB} + x_{BA} \)

\[
\frac{dx_{PA}}{dt} = \frac{dx_{PB}}{dt} + \frac{dx_{BA}}{dt} \implies v_{PA} = v_{PB} + v_{BA}
\]
\[
\frac{dv_{PA}}{dt} = \frac{dv_{PB}}{dt} + \frac{dv_{BA}}{dt} \quad \Rightarrow \quad a_{PA} = a_{PB} + a_{BA}
\]

- **Note:** if \(a_{BA} = 0\) then \(a_{PA} = a_{PB}\).

Different observers moving at constant velocity relative to each other will measure the same acceleration for a moving particle \(P\).
Two cars $A$ and $B$ move on straight line with constant velocities $v_A = 20 \text{ km/h}$ and $v_B = -30 \text{ km/h}$ relative to a stationary observer. Recall that $v_{BA}$ is the velocity of $B$ measured by the driver $A$ while $v_{AB}$ is the velocity of $A$ measured by the driver $B$.

\[
\begin{align*}
\text{A)} & \quad v_{AB} = 20 \text{ km/h}, \quad v_{BA} = -30 \text{ km/h} \\
\text{B)} & \quad v_{AB} = 50 \text{ km/h}, \quad v_{BA} = -50 \text{ km/h} \\
\text{C)} & \quad v_{AB} = 50 \text{ km/h}, \quad v_{BA} = 50 \text{ km/h} \\
\text{D)} & \quad v_{AB} = -50 \text{ km/h}, \quad v_{BA} = -50 \text{ km/h} \\
\text{E)} & \quad v_{AB} = -30 \text{ km/h}, \quad v_{BA} = 20 \text{ km/h}
\end{align*}
\]
**Answer**

Two cars $A$ and $B$ move on straight line with constant velocities $v_A = 20\, \text{km/h}$ and $v_B = -30\, \text{km/h}$ relative to a stationary observer. Recall that $v_{BA}$ is the velocity of $B$ measured by the driver $A$ while $v_{AB}$ is the velocity of $A$ measured by the driver $B$.

\[ v_A = 20\, \text{km/h} \quad v_B = -30\, \text{km/h} \]

\[ v_A = v_B + v_{AB} \quad v_B = v_A + v_{BA} \]

- **A)** $v_{AB} = 20\, \text{km/h}$, $v_{BA} = -30\, \text{km/h}$
- **B)** $v_{AB} = 50\, \text{km/h}$, $v_{BA} = -50\, \text{km/h}$
- **C)** $v_{AB} = 50\, \text{km/h}$, $v_{BA} = 50\, \text{km/h}$
- **D)** $v_{AB} = -50\, \text{km/h}$, $v_{BA} = -50\, \text{km/h}$
- **E)** $v_{AB} = -30\, \text{km/h}$, $v_{BA} = 20\, \text{km/h}$
The motion of particle $P$ is studied by two observers moving at constant velocity relative to each other. Observer $B$ is moving with constant velocity $\vec{v}_{BA}$ relative to observer $A$. 

Fig. 4-19  Frame $B$ has the constant two-dimensional velocity $\vec{v}_{BA}$ relative to frame $A$. The position vector of $B$ relative to $A$ is $\vec{r}_{BA}$. The position vectors of particle $P$ are $\vec{r}_{PA}$ relative to $A$ and $\vec{r}_{PB}$ relative to $B$. 

Relative motion in two dimensions
\[ \vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA} \]

• \( \vec{r}_{PA} \) position vector of \( P \) relative to \( A \)

• \( \vec{r}_{PB} \) position vector of \( P \) relative to \( B \)

• \( \vec{r}_{BA} \) position vector of \( B \) relative to \( A \)

• same notation for velocity and acceleration
\[ \vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA} \]

\[ \downarrow \quad \frac{d}{dt} \]

\[ \vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA} \]

\[ \downarrow \quad \frac{d}{dt} \]

\[ \vec{a}_{PA} = \vec{a}_{PB} + \vec{a}_{BA} \]

\[ \vec{a}_{BA} = 0 \quad \Rightarrow \quad \vec{a}_{PA} = \vec{a}_{PB} \]

**Fig. 4-19** Frame B has the constant two-dimensional velocity \( \vec{v}_{BA} \) relative to frame A. The position vector of B relative to A is \( \vec{r}_{BA} \). The position vectors of particle P are \( \vec{r}_{PA} \) relative to A and \( \vec{r}_{PB} \) relative to B.
Problem 75 Chapter 4

A train travels due south at 30 m/s (relative to the ground) in a rain that is blown toward the south by the wind. The path of each raindrop makes an angle $\theta = 70^\circ$ with the vertical, as measured by an observer stationary on the ground. An observer on the train, however, sees the drops fall perfectly vertically. Determine the speed of the raindrops relative to the ground.
\[ \vec{v}_{\text{train}} = v_{\text{train}} \hat{i} \]

\[ \vec{v}_{\text{rel}} = -v_{\text{rel}} \hat{j} \]

\[ \vec{v}_{\text{drop}} = \vec{v}_{\text{train}} + \vec{v}_{\text{rel}} \]

\[ \sin(\theta) = \frac{v_{\text{train}}}{v_{\text{drop}}} \]

\[ v_{\text{drop}} = \frac{v_{\text{train}}}{\sin(\theta)} \]

\[ = 31.92533316 \ldots \text{ m/s} \]

\[ \tan(\theta) = \frac{v_{\text{train}}}{v_{\text{rel}}} \]

\[ v_{\text{rel}} = \frac{v_{\text{train}}}{\tan(\theta)} \]

\[ = 10.91910700 \ldots \text{ m/s} \]
\[ \vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j} \quad \text{relative to sled} \]

1) Find \( v_{0x}, v_{0y} \) assuming the ball lands at the same height as launching point.
Relative to ground:
\[ \vec{v}_{bg,0} = \vec{v}_0 + \vec{v}_s = (v_{0x} - v_s)i + v_{0y}j \]

Total flying time
\[ \Delta t = \frac{2v_{0y}}{g} \]

\[ \Delta x_{bg} = (v_{0x} - v_s)\Delta t = \frac{2v_{0x}v_{0y}}{g} - \frac{2v_{0y}v_s}{g} \]

**Magnitude** of slope of the line
\[ \frac{2v_{0y}}{g} = 4 \Rightarrow v_{0y} = 2g = 19.6 \text{ m/s} \]

Intersection with vertical axis
\[ v_s = 0 \Rightarrow \frac{2v_{0x}v_{0y}}{g} = 40 \]
\[ v_{0x} = 10 \text{ m/s} \]
2) Find horizontal displacement relative to sled, assuming the speed of the sled does not change when the ball is shot.

In reference frame of the sled

\[ \vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j} \]

Flying time the same

\[ \Delta t = \frac{2v_{0y}}{g} \]

\[ \Delta x = v_{0x}\Delta t = \frac{2v_{0y}v_{0x}}{g} \]