Rutgers University
Department of Physics & Astronomy

01:750:271 Honors Physics I

Lecture 3
Midterm 1

• Sunday October 8th 12:00pm
• Physics Lecture Hall
• Lectures 1-4 (one dim motion – 2d and 3d motion, relative motion)
• No forces!
3. Vectors

- **Vectors**: quantities which indicate **both** magnitude and direction.
  
  **Examples**: displacement, velocity, acceleration

- **Scalars**: quantities which indicate **only** magnitude.
  
  **Examples**: time, speed, mass
• **Vectors** are represented by **arrows**:

  (i) The length of the arrow signifies magnitude.

  (ii) The head of the arrow signifies direction.

  **Displacement vector** for a particle travelling from \( A \) to \( B \) on a straight path

  **Note:** All three vectors are identical because they have the same direction and magnitude.

  A shift preserving both direction and magnitude does not change the vector. (**Translation.**
Displacement vector for a particle travelling on a curved path.

**Note:** independent of the path from $A$ to $B$.

- **Notation:**

  $a, b, c, \ldots$  or  $\vec{a}, \vec{b}, \vec{c}, \ldots$

  The **magnitude** of a vector $\vec{a}$:

  $a$  or  $|\vec{a}|$
Adding vectors geometrically

• What is the **sum** of two vectors?

\[ \vec{a} + \vec{b} = ? \]

• **Step 1.** Draw the vectors **head to tail**
Step 2. The vector sum of \( \vec{a} \) and \( \vec{b} \) is the vector \( \vec{c} \) pointing from the tail of \( \vec{a} \) to the head of \( \vec{b} \).

Mathematical formula:

\[
\vec{a} + \vec{b} = \vec{c}
\]
• **Commutativity:** \( \vec{a} + \vec{b} = \vec{b} + \vec{a} \)
• **Associativity:** \((\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})\)
• **Inverse:** \( \vec{a} + ( - \vec{a} ) = 0 \)

  ![Diagram of vector addition and subtraction]

  **Note:** \(-\vec{a}\) has the same magnitude as \(\vec{a}\), but it points in opposite direction.

• **Vector subtraction:** \( \vec{a} - \vec{b} = \vec{a} + ( -\vec{b} ) \).

  ![Diagram of vector subtraction]

  \( \vec{a} - \vec{b} \)

  \( \vec{a} \)

  \( \vec{b} \)

  \( \vec{a} + \vec{b} \)
Multiplying vectors by scalars

• If \( \vec{a} \) vector, \( s \neq 0 \) number then

\[
s\vec{a} = \text{vector with magnitude } |s\vec{a}| = s|\vec{a}|
\]

and direction \( \begin{cases} 
\text{same as } \vec{a} & \text{if } s > 0 \\
\text{opposite to } \vec{a} & \text{if } s < 0 
\end{cases} \)

Example

\[
1.75 \vec{a} \quad \text{Note:} \quad 0 \vec{a} = 0.
\]

\[
-0.8 \vec{a}
\]
Which of the following statements is **false** for the three vectors below?

\[ \vec{a} + \vec{b} + \vec{c} = 0 \]

\[ \vec{c} + \vec{b} = -\vec{a} \]

\[ |\vec{c}| < |\vec{a}| + |\vec{b}| \]

\[ |\vec{c}| = |\vec{a}| + |\vec{b}| \]

**E) None of the above.**
Answer

Which of the following statements is false for the three vectors below?

- **A)** $\vec{a} + \vec{b} + \vec{c} = 0$
- **B)** $\vec{c} + \vec{b} = -\vec{a}$
- **C)** $|\vec{c}| < |\vec{a}| + |\vec{b}|$
- **D)** $|\vec{c}| = |\vec{a}| + |\vec{b}|$
- **E)** None of the above.

Triangle inequality: $|\vec{c}| < |\vec{a}| + |\vec{b}|$ since $\vec{a}, \vec{b}, \vec{c}$ not colinear.
Which of the following relations is true for the three vectors below?

\[ \vec{a} + \vec{b} + \vec{c} = 0 \]  
\[ \vec{a} + \vec{b} - \vec{c} = 0 \]  
\[ \vec{a} + \vec{b} - 2\vec{c} = 0 \]  
\[ \vec{a} + \vec{b} + 2\vec{c} = 0 \]  
\[ \text{E) None of the above.} \]
Which of the following relations is true for the three vectors below?

\[ A) \, \vec{a} + \vec{b} + \vec{c} = 0 \]
\[ B) \, \vec{a} + \vec{b} - \vec{c} = 0 \]
\[ C) \, \vec{a} + \vec{b} - 2\vec{c} = 0 \]
\[ D) \, \vec{a} + \vec{b} + 2\vec{c} = 0 \]
\[ E) \, \text{None of the above.} \]
i-Clicker

Which of the following relations is true for the three vectors below?

\[ \vec{a} + \vec{b} + \vec{c} = 0 \]

\[ \vec{a} + \vec{b} - \vec{c} = 0 \]

\[ \vec{a} + \vec{b} - 2\vec{c} = 0 \]

\[ \vec{a} + \vec{b} + 2\vec{c} = 0 \]

\[ \text{E) None of the above.} \]
i-Clicker

Which of the following relations is true for the three vectors below?

![Vectors diagram]

A) \( \vec{a} + \vec{b} + \vec{c} = 0 \)

B) \( \vec{a} + \vec{b} - \vec{c} = 0 \)

C) \( \vec{a} + \vec{b} - 2\vec{c} = 0 \)

D) \( \vec{a} + \vec{b} + 2\vec{c} = 0 \)

E) None of the above.
Components of vectors

- **Axis** = line equipped with a *preferred direction*, also called *orientation*.

**Example**: one dimensional motion

- **positive direction** $x \rightarrow$

  $\bigcirc = \text{origin}: \ x = 0$
• **Projection:** suppose \( \vec{a} \) and a given axis are in the same plane

Note:
• \( \vec{a}_{\text{proj}} \) is a vector along the given axis.
• \( \vec{a}_{\text{proj}} \) is **not** the component of \( \vec{a} \) along the given axis.

(as stated in the textbook.)
• The component of $\vec{a}$ along given axis is a number

$$a_\parallel = \begin{cases} 
|\vec{a}_{\text{proj}}| & \text{if } \vec{a}_{\text{proj}} \text{ points in the positive direction} \\
-|\vec{a}_{\text{proj}}| & \text{if } \vec{a}_{\text{proj}} \text{ points in the negative direction}
\end{cases}$$

$$a_\parallel = |\vec{a}_{\text{proj}}| > 0$$

$$a_\parallel = -|\vec{a}_{\text{proj}}| < 0$$
• Right triangle rule

\[ a_{\parallel} = a \cos \theta \]

\[ a_{\parallel} = \left| \overrightarrow{a}_{\text{proj}} \right| = a \cos \theta \]

\[ \theta = \text{angle between the axis and the vector (counterclockwise)} \]
\[ |\vec{a}_{\text{proj}}| = a \cos (\theta - \pi) = -a \cos \theta \]
\[ a_{||} = -|\vec{a}_{\text{proj}}| = a \cos \theta \]

\( \theta = \text{angle between the axis and the vector (counterclockwise)} \)
Summary:

- The **projection** of $\vec{a}$ is the **vector** $\vec{a}_{\text{proj}}$.
- The **component** of $\vec{a}$ is the **number**
  \[ a_{\parallel} = a \cos \theta \]
• **Right handed coordinate system:** three **mutually orthogonal** axes meeting at a point O called **origin**.

\[90^\circ = \frac{\pi}{2}\]

- The \(x\) and \(y\) axes are in the page.
- The \(z\)-axis sticks out of the page.

\(x, y, z\): **coordinates**
• The **components** of \( \vec{a} \) along the three axes

\[ \vec{a}_1, \vec{a}_2, \vec{a}_3: \text{ the projections of } \vec{a} \text{ on the } x, y, z \text{ axes. (vectors)} \]

\[ a_x, a_y, a_z: \text{ the **components** of } \vec{a} \text{ along the } x, y, z \text{ axes (numbers)} \]
• **Planar vectors** in $x, y$ plane

![Diagram showing planar vectors](image)

- This is the $y$ component of the vector.
- This is the $x$ component of the vector.
- The components and the vector form a right triangle.
The **right triangle rules** for planar vectors

\[
\begin{align*}
    a_x &= a \cos \theta \\
    a_y &= a \sin \theta \\
    a &= \sqrt{a_x^2 + a_y^2} \\
    \tan \theta &= \frac{a_y}{a_x} \\
    (\text{if } a_x \neq 0).
\end{align*}
\]
Vector Addition 2.02

| R | 12.0 | θ | 48.4 | Rx | 8 | Ry | 9 |

Grab one

Show Sum

Component Display
- None
- Style 1
- Style 2
- Style 3

Show Grid

Clear All

PhET
i-Clicker

A vector $\vec{a}$ is contained in the $(y, z)$ plane such that the angle between $\vec{a}$ and the $y$ axis is $\phi$. What are the components of $\vec{a}$?

A) $a_x = \cos \phi$, $a_y = \sin \phi$, $a_z = 0$

B) $a_x = \cos \phi$, $a_y = 0$, $a_z = \sin \phi$

C) $a_x = 0$, $a_y = \sin \phi$, $a_z = \cos \phi$

D) $a_x = 0$, $a_y = \cos \phi$, $a_z = \sin \phi$

E) $a_x = \sin \phi$, $a_y = 0$, $a_z = \cos \phi$
Answer

A vector \( \vec{a} \) is contained in the \((y, z)\) plane such that the angle between \( \vec{a} \) and the \( y \) axis is \( \phi \). What are the components of \( \vec{a} \)?

A) \( a_x = \cos \phi, \ a_y = \sin \phi, \ a_z = 0 \)
B) \( a_x = \cos \phi, \ a_y = 0, \ a_z = \sin \phi \)
C) \( a_x = 0, \ a_y = \sin \phi, \ a_z = \cos \phi \)
D) \( a_x = 0, \ a_y = \cos \phi, \ a_z = \sin \phi \)
E) \( a_x = \sin \phi, \ a_y = 0, \ a_z = \cos \phi \)
**Unit vectors**

- **Unit vector** = vector of magnitude 1 pointing in the positive direction along an axis

\[ \hat{u} \quad |\hat{u}| = 1 \]
• **Unit vectors** for a right handed coordinate system

If \( \vec{a} \) has **components** \( a_x, a_y, a_z \), its **projections** are

\[
\vec{a}_1 = a_x \hat{i} \\
\vec{a}_2 = a_y \hat{j} \\
\vec{a}_3 = a_z \hat{k}
\]

\[
\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}
\]
• Two vectors are equal if and only if their components are equal.

\[ \vec{a} = \vec{b} \iff a_x = b_x, \ a_y = b_y, \ a_z = b_z. \]
Which of the following expressions is correct for the vector $\vec{a}$ shown below?

$A) \quad \vec{a} = a \cos \phi \hat{i} + a \sin \phi \hat{j}$

$B) \quad \vec{a} = a \sin \phi \hat{i} + a \cos \phi \hat{j}$

$C) \quad \vec{a} = -a \sin \phi \hat{i} + a \cos \phi \hat{j}$

$D) \quad \vec{a} = a \cos \phi \hat{i} - a \sin \phi \hat{j}$

$E) \quad$ None of the above.
Answer

Which of the following expressions is correct for the vector \( \vec{a} \) shown below?

\[ A) \quad \vec{a} = a \cos \phi \hat{i} + a \sin \phi \hat{j} \]
\[ B) \quad \vec{a} = a \sin \phi \hat{i} + a \cos \phi \hat{j} \]
\[ C) \quad \vec{a} = -a \sin \phi \hat{i} + a \cos \phi \hat{j} \]
\[ D) \quad \vec{a} = a \cos \phi \hat{i} - a \sin \phi \hat{j} \]
\[ E) \quad \text{None of the above.} \]
Adding vectors by components

- For any two vectors:
  \[ \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \quad \vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k} \]

  we have:
  \[ \vec{a} + \vec{b} = (a_x + b_x) \hat{i} + (a_y + b_y) \hat{j} + (a_z + b_z) \hat{k} \]
  \[ \vec{a} - \vec{b} = (a_x - b_x) \hat{i} + (a_y - b_y) \hat{j} + (a_z - b_z) \hat{k} \]

  More generally, if \( s, t \) are scalars,
  \[ s\vec{a} + t\vec{b} = (sa_x + tb_x) \hat{i} + (sa_y + tb_y) \hat{j} + (sa_z + tb_z) \hat{k} \]
The scalar product

Associates to any two vectors $\vec{a}, \vec{b}$ the number

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \phi$$

$$= \vec{b} \cdot \vec{a}$$

commutative

Order is irrelevant!
• Scalar product in unit vector notation

\[ \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \]

\[ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \]

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \quad \vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k} \]

\[ \vec{a} \cdot \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \cdot (b_x \hat{i} + b_y \hat{j} + b_z \hat{k}) \]

\[ = a_x b_x + a_y b_y + a_z b_z \]

• Scalar product and components

\[ \vec{a} = (\vec{a} \cdot \hat{i}) \hat{i} + (\vec{a} \cdot \hat{j}) \hat{j} + (\vec{a} \cdot \hat{k}) \hat{k} \]

\[ a_x = \vec{a} \cdot \hat{i}, \quad a_y = \vec{a} \cdot \hat{j}, \quad a_z = \vec{a} \cdot \hat{k} \]
• **Example:** the magnitude of vector sum and difference

\[
|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})
\]
\[
= \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}
\]
\[
= |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos(\theta)
\]

\[
|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})
\]
\[
= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}
\]
\[
= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\theta)
\]
Vectors and the laws of physics

- Relations among vectors do not depend on the choice of a coordinate system.

- Relations in physics are also independent of the choice of a coordinate system.

\[
a = \sqrt{a_x^2 + a_y^2} = \sqrt{(a'_x)^2 + (a'_y)^2} \quad \theta = \theta' + \phi
\]
Position and Displacement

- In **3D** the position of a particle is given by a **position vector**

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \]

starting at the origin \( O \).

- \( x, y, z \) are the **components** of \( \vec{r} \), also called the **coordinates** of the particle.

- The path of the particle is generally a **curve**.
• **Displacement:** the change of the position vector $\vec{r}$ over a time interval $\Delta t$.

![Diagram showing displacement vector $\Delta \vec{r}$ with initial position $\vec{r}_1$ and final position $\vec{r}_2$.]

\[ \vec{r}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} \quad \text{at time } t_1 \]
\[ \vec{r}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \quad \text{at time } t_2 \]

\[ \Delta \vec{r} = (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) \]
\[ = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k} \]
\[ = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k}. \]
A particle travels along a 3D path as shown in the figure starting from the initial position \( \vec{r}_1 = (1 \text{ m})\hat{i} \) and ending with the final position \( \vec{r}_2 = (0.5 \text{ m})\hat{i} \). What is the displacement vector?

\[ \mathbf{A}) \quad \Delta \vec{r} = (-1 \text{ m})\hat{i} + (0.7 \text{ m})\hat{j} \]
\[ \mathbf{B}) \quad \Delta \vec{r} = (0.7 \text{ m})\hat{j} - (0.3 \text{ m})\hat{k} \]
\[ \mathbf{C}) \quad \Delta \vec{r} = (-0.5 \text{ m})\hat{i} \]
\[ \mathbf{D}) \quad \Delta \vec{r} = 0 \]
\[ \mathbf{E}) \quad \Delta \vec{r} = (0.5 \text{ m})\hat{i} \]
A particle travels along a 3D path as shown in the figure starting from the initial position \( \vec{r}_1 = (1 \text{ m})\hat{i} \) and ending with the final position \( \vec{r}_2 = (0.5 \text{ m})\hat{i} \). What is the displacement vector?

\[
\begin{align*}
A) \quad & \Delta \vec{r} = (-1 \text{ m})\hat{i} + (0.7 \text{ m})\hat{j} \\
B) \quad & \Delta \vec{r} = (0.7 \text{ m})\hat{j} - (0.3 \text{ m})\hat{k} \\
C) \quad & \Delta \vec{r} = (-0.5 \text{ m})\hat{i} \\
D) \quad & \Delta \vec{r} = 0 \\
E) \quad & \Delta \vec{r} = (0.5 \text{ m})\hat{i}
\end{align*}
\]
Average and Instantaneous Velocity

- **Average velocity** over a time interval $\Delta t$

$$\vec{v}_{\text{avg}} = \frac{\text{Displacement vector}}{\text{time interval}} = \frac{\Delta \vec{r}}{\Delta t}$$

**Note:**

Since $\Delta t > 0 \Rightarrow \vec{v}_{\text{avg}}$ is always parallel with $\Delta \vec{r}$ and points in the same direction.
• **Instantaneous velocity** at time $t$

\[ \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \left( \frac{dx}{dt} \right) \hat{i} + \left( \frac{dy}{dt} \right) \hat{j} + \left( \frac{dz}{dt} \right) \hat{k} \]

**Note:**

• $\vec{v}$ is always **tangent** to the path of the particle at its current position.
• **Components** of $\vec{v}$:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$
Example: circular trajectory

\[ v_x = v \sin(\theta), \quad v_y = -v \cos(\theta) \]
Average and Instantaneous Acceleration

- **Average acceleration** over a time interval $\Delta t$

$$\vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t}$$

**Note:**

Since $\Delta t > 0 \Rightarrow \vec{a}_{\text{avg}}$ is always parallel with $\Delta \vec{v}$ and points in the same direction.
• **Instantaneous acceleration** at time $t$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \left( \frac{dv_x}{dt} \right)\hat{i} + \left( \frac{dv_y}{dt} \right)\hat{j} + \left( \frac{dv_z}{dt} \right)\hat{k}$$

• **Acceleration** is a vector which measures the change of the velocity vector.

• **Components** of $\vec{a}$:

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}$$
• The acceleration is non-zero if
  
  either

  the magnitude of \( \vec{v} \) changes.

  or

  the direction of \( \vec{v} \) changes.

• The acceleration is 0 if and only if both the magnitude and the direction of \( \vec{v} \) are constant.
A particle moves on a path as shown below such that the **magnitude** of its velocity vector is **constant**. Where is the instantaneous acceleration \( \vec{a} \) zero?

- **A)** Everywhere
- **B)** At \( A \)
- **C)** At \( C \)
- **D)** At \( B \)
- **E)** Nowhere
Answer

A particle moves on a path as shown below such that the **magnitude** of its velocity vector is **constant**. Where is the instantaneous acceleration \( \vec{a} \) zero?

<table>
<thead>
<tr>
<th>Option</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>A)</td>
<td>Everywhere</td>
</tr>
<tr>
<td>B)</td>
<td>At A</td>
</tr>
<tr>
<td>C)</td>
<td>At C</td>
</tr>
<tr>
<td>D)</td>
<td>At B</td>
</tr>
<tr>
<td>E)</td>
<td>Nowhere</td>
</tr>
</tbody>
</table>

\[ |\vec{v}| \text{ constant } \Rightarrow \vec{v} \text{ constant on linear segment; } \vec{v} \text{ not constant on curved segments; changes direction.} \]
A particle moves on a helix as shown below such that the $y$-component of its velocity is constant. Which of the following statements is **false**?

- $A) \ a_x \neq 0$
- $B) \ a_z \neq 0$
- $C) \ a_y = 0$
- $D) \ a_y \neq 0$
- $E) \ None \ of \ the \ above.$
Answer

A particle moves on a helix as shown below such that the $y$-component of its velocity is constant. Which of the following statements is false?

A) $a_x \neq 0$

B) $a_z \neq 0$

C) $a_y = 0$

D) $a_y \neq 0$

E) None of the above.
Example

A bus is moving horizontally with constant speed \( v = 10 \text{ m/s} \). Inside the bus a ball falls a distance \( h = 1 \text{ m} \) in uniform gravitational field with zero initial velocity. Assuming air resistance negligible, what is the horizontal displacement of the ball during the fall *relative to the ground*? What is the velocity vector of the ball when it hits the floor measured by a ground based observer?
Falling time:

\[ y = h - gt^2/2 \quad \Rightarrow \quad \Delta t = \sqrt{\frac{2h}{g}} \]

Horizontal displacement:

\[ \Delta x = v\Delta t = v\sqrt{\frac{2h}{g}} \]

\[ v_y = -gt, \quad v_x = v \]

Final velocity vector:

\[ \vec{v}_f = \vec{v}_i - \sqrt{2g}h \hat{j} \]