Rutgers University
Department of Physics & Astronomy

01:750:123H Honors Analytical Physics I

Lecture 2
One dimensional motion.

- Motion is along a straight line only.
- Physical objects will be assumed pointlike.
• One dimensional motion ↔ graph of the position $x$ as function of time $t$
• **Displacement:** change from position \( x_1 \) to \( x_2 \)

\[ \Delta x = x_2 - x_1 \]

• **Average velocity:** rate of change of position

\[ v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \]

• **Average speed**

\[ s_{\text{avg}} = \frac{\text{total distance travelled in time interval } \Delta t}{\Delta t} \]

• \( \Delta x \), \( v_{\text{avg}} \) **vectors** (sign and magnitude \( \geq 0 \))

• \( s_{\text{avg}} \) **scalar** (magnitude \( \geq 0 \))
Instantaneous velocity and speed

- **Instantaneous velocity**: velocity of a particle at a given moment in time.

\[ v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \]

\[ v = \text{limit of } v_{\text{avg}} \text{ over smaller and smaller time intervals } \Delta t \text{ centered at a current point } (x, t) \]
Geometric interpretation

\[ v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \]

= slope of the tangent line to motion graph at current point

\[ = \frac{dx}{dt} \text{ (derivative)} \]
Note: $\mathbf{v}$ is a vector quantity \( \{ \) direction \( \} \) magnitude

- **Instantaneous speed:** magnitude of $\mathbf{v}$

\[
s = |\mathbf{v}| = \left| \frac{dx}{dt} \right| \geq 0
\]

Units for $\mathbf{v}, s$: m/s.
**Acceleration**

- **Acceleration**: the rate of change of velocity.

- **Average acceleration**

  \[
  a_{avg} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}
  \]

  \[v_1 = v(t_1) \text{ instantaneous velocity at time } t_1\]

  \[v_2 = v(t_2) \text{ instantaneous velocity at time } t_2 > t_1\]

  \[a_{avg} \text{ vector quantity: same sign as } \Delta v \text{ since } t_2 - t_1 > 0\]
• **Instantaneous acceleration**

\[ a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \]

**Note:** \( a \) is a vector quantity \( \left\{ \begin{array}{l} \text{direction} \\ \text{magnitude} \end{array} \right. \)

**Units for** \( a_{\text{avg}}, a \) :

\[ \text{(meter/second)/second} = \text{m/s}^2. \]
**Geometric interpretation:** velocity graph

\[
a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}
\]

- $a$ = slope of the tangent line to motion graph at current point
- $\frac{dv}{dt}$

Alternative formula:

\[
a = \frac{d^2x}{dt^2} \quad \text{(second derivative)}
\]
**Constant acceleration**

What if $a$ is constant, independent of time?

Time dependence of velocity $v$ and position $x$?

$$a = \text{constant} \quad \Rightarrow \quad a = a_{\text{avg}}$$

$a_{\text{avg}}$ = average acceleration from $t = 0$ to time $t > 0$:

$$a_{\text{avg}} = \frac{v - v_0}{t - 0} = \frac{v - v_0}{t}, \quad v_0 = \text{velocity at } t = 0$$

$$v = v_0 + at \quad \text{Linear!}$$
Average velocity from $t = 0$ to time $t$:

$$v \quad \text{Linear} \quad \Rightarrow \quad v_{\text{avg}} = \frac{v + v_0}{2} = v_0 + \frac{at}{2}$$

By definition

$$v_{\text{avg}} = \frac{x - x_0}{t - 0} = \frac{x - x_0}{t}, \quad x_0 = \text{position at } t = 0.$$ 

$$x = x_0 + v_0 t + \frac{at^2}{2}$$
Third equation $v \leftrightarrow x$

$$v = v_0 + at \quad \Rightarrow \quad at = v - v_0$$

$$x = x_0 + v_0 t + at^2/2 \quad \Rightarrow \quad ax = ax_0 + v_0(at) + (at)^2/2$$

Substitution:

$$ax = ax_0 + v_0(v - v_0) + (v - v_0)^2/2$$  No t here!

Algebra: $(v - v_0)^2 = v^2 + v_0^2 - 2vv_0$. Then

$$v^2 = v_0^2 + 2a(x - x_0)$$
\[ x = x_0 + v_0 t + \frac{at^2}{2} \]

\[ v = v_0 + at \]

\[ a = \text{constant} \]
Another useful formula:

\[ v^2 = v_0^2 + 2a(x - x_0) \]

**Warning!**

The above equations are **not** valid if \( a \neq \text{constant} \).
For general motion with \( a \neq \text{constant} \):

\[
v = \int a(t)dt + c, \quad x = \int v(t)dt + c'
\]

\( \int f(t)dt = \text{integral (anti-derivative) of } f(t) \)

\( c, c' \) constants determined from initial conditions

\[
v(0) = v_0, \quad x(0) = x_0.
\]
Example. Suppose \( a(t) = 1.5t \) and \( v_0 = -1 \ m/s \), \( x_0 = 2 \ m \). How do we determine \( v(t) \) and \( x(t) \)?

\[
a = \frac{dv}{dt} \Rightarrow v = \int a(t)dt + c_1 = 1.5(t^2/2) + c_1 = 0.75t^2 + c_1
\]

where \( c_1 \) is an unknown constant.

\[
v(0) = -1 \ m/s \Rightarrow c_1 = -1 \ m/s \Rightarrow v(t) = 0.75t^2 - 1
\]

\[
v = \frac{dx}{dt} \Rightarrow x = \int v(t)dt + c_2
\]

where \( c_2 \) is another unknown constant.

\[
x(t) = \int (0.75t^2-1)dt + c_2 = 0.75(t^3/3) - t + c_2 = 0.25t^3 - t + c_2
\]

\[
x(0) = 2 \ m \Rightarrow c_2 = 2 \ m \Rightarrow x(t) = 0.25t^3 - t + 2.
\]
**Problem 35, Ch. II**

A red car and a green car move toward each other.

Initially at $t = 0$:

- $x_{g0} = 270 \text{ m}$
- $x_{r0} = -60 \text{ m}$
- $v_{g0} = -20 \text{ m/s}$
- $v_{r0} = 0 \text{ m/s}$

Know that $v_g, a_r$ are constant. Then $a_r =$?
\( v_g = v_{g0} = \text{constant} \Rightarrow \)

\[ x_g = x_{g0} + v_{g,0}t \]

\( a_r = \text{constant}, \ v_{r0} = 0 \Rightarrow \)

\[ x_r = x_{r0} + a_r t^2 / 2 \]

Know that \( x_g = x_r \) at \( t = 12 \text{ s} \Rightarrow \)

\[ x_{g0} + v_{g,0}t = x_{r0} + a_r t^2 / 2 \]

Hence

\[ a_r = \frac{x_{g0} + v_{g,0}t - x_{r0}}{t^2 / 2} \]

\[ = \frac{270 - 20 \times 12 + 60}{72} \]

\[ = 1.25 \text{ m/s}^2. \]
Another example

Suppose car $A$ starts moving from the origin at $t = 0$ with constant constant acceleration $a_A = 1 \text{ m/s}^2$ and initial velocity $v_{0A} = 0 \text{ m/s}$. Car $B$ starts moving from the origin at time $t = 2 \text{ s}$ with constant acceleration $a_B$ and initial velocity $v_{0B} = 0$. What is $a_B$ if $B$ catches up with $A$ at $x = 72 \text{ m}$?
Suppose car A starts moving from the origin at $t = 0$ with constant constant acceleration $a_A = 1 \text{ m/s}^2$ and initial velocity $v_{0A} = 0 \text{ m/s}$. Car B starts moving from the origin at time $t = 2 \text{ s}$ with constant acceleration $a_B$ and initial velocity $v_{0B} = 0$. What is $a_B$ if $B$ catches up with $A$ at $x = 72 \text{ m}$?

\[
x_A = x_{0A} + v_{0A}t + a_A t^2/2 = a_A t^2/2
\]
\[
x_B = x_{0B} + v_{0B}(t - 2) + a_B(t - 2)^2/2 = a_B(t - 2)^2/2
\]

$x_A = 72$, $a_A = 1 \Rightarrow t = 12 \text{ s}$

$x_B = 72$, $t = 12 \Rightarrow a_B = \frac{x_B}{(t - 2)^2/2} = 1.44 \text{ m/s}^2$
Free-fall acceleration

- **Free-fall**: motion of objects close to Earth’s surface in absence of all external forces except for their weight.

- In **vacuum** all objects accelerate downwards at the **same constant rate**.

\[ a_{\text{apple}} = a_{\text{feather}} \]
• **Constant acceleration model**

![Diagram showing constant acceleration model](image)

- $y = -$ height with respect to Earth’s surface.
- $a = -g = -9.8 \text{ m/s}^2$ for all objects if air resistance is negligible.
- $g = 9.8 \text{ m/s}^2$ magnitude of acceleration.

\[
\begin{align*}
  v &= v_0 - gt \\
  y &= y_0 + v_0 t - gt^2 / 2 \\
  v^2 &= v_0^2 - 2g(y - y_0)
\end{align*}
\]
Problem 59, Ch. II

Water drips from the nozzle of a shower onto the floor 200 cm below. The drops fall at regular (equal) intervals of time, the first drop striking the floor at the instant the fourth drop begins to fall. When the first drop strikes the floor, how far below the nozzle are the (a) second and (b) third drops?
Total time for the 1st drop to hit the floor:

\[ y = h - gt^2/2 \]

\[ y = 0 \Rightarrow t_1 = \frac{\sqrt{2h/g}}{2} \quad (= 0.6388765650 \, s) \]

When the 1st hits the floor the second will have fallen for

\[ t_2 = \frac{2t_1}{3} = \frac{2}{3} \sqrt{2h/g} \quad (= 0.4259177100 \, s) \]

Hence

\[ h - y_2 = gt_2^2/2 = \frac{g}{2} \times \frac{4}{9} \times \frac{2h}{g} = \frac{4h}{9} \quad (= 0.8888888889 \, m) \]
When the 1st hits the floor the third will have fallen for

\[ t_2 = t_1/3 = \frac{1}{3} \sqrt{\frac{2h}{g}} \quad (= 0.2129588550 \text{ s}) \]

Hence

\[ h - y_3 = gt_2^2/2 = \frac{g}{2} \times \frac{1}{9} \times \frac{2h}{g} = \frac{h}{9} = 0.2222222222 \text{ m} \]
Problem 62, Ch. II

A basketball player grabbing a rebound jumps 76.0 cm vertically. How much total time (ascent and descent) does the player spend (a) in the top 15.0 cm of this jump and (b) in the bottom 15.0 cm? Do your results explain why such players seem to hang in the air at the top of a jump?

**Ascent:**

\[ v^2 = v_0^2 - 2gy \]

\[ y = h \Rightarrow v = 0, \quad v_0 = \sqrt{2gh} \]

\[ v(y) = \sqrt{v_0^2 - 2gy} = \sqrt{2g(h - y)} \]
Time from $y = 0$ to $y = 0.15$ m

$v - v_0 = -gt_1 \Rightarrow t_1 = \frac{v_0 - v}{g} = \sqrt{\frac{h}{2g}} - \sqrt{\frac{h - y}{2g}} = 0.0204994246 \text{ s.}$

Time from $y = h - 0.15 = 0.61$ m to $y = h = 0.76$ m

$0 - v = -gt_2 \Rightarrow t_2 = \frac{v}{g} = \sqrt{\frac{h - y}{2g}} = 0.1764155576 \text{ s.}$
Descent:

\[ v^2 = 2g(h - y) \implies |v_f| = \sqrt{2gh} = v_0 \]

\[ v_f = -v_0, \quad v = -\sqrt{2g(h - y)} \]

as vectors. Time from \( y = h \) to \( y = h - 0.15 = 0.61 \)

\[ v - 0 = -gt_3 \implies t_3 = \frac{|v|}{g} \]

\[ = \sqrt{\frac{h - y}{2g}} \]

\[ = 0.1764155576 \ s = t_2 \]

Time from \( y = 15 \text{ cm} \) to \( y = 0 \text{ cm} \)

\[ t_4 = \sqrt{\frac{h}{2g}} - \sqrt{\frac{h - y}{2g}} = 0.0204994246 \ s = t_1 \]
Total time in bottom 15cm

\[2 \times t_1 = 0.0409988492 \ s\]

Total time in top 15cm

\[2 \times t_2 = 0.3528311152 \ s\]

Ratio:

\[\frac{t_2}{t_1} = 8.605878508.\]