13. Gravitation

Newton’s law of gravitation

Every point particle attracts every other particle with a gravitational force

\[ F = G \frac{m_1 m_2}{r^2} \]

\[ G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 = 6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \]
Shell theorem

A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell’s mass were concentrated at its center.

\[ F = G \frac{mM}{r^2} \]

- \( m \) mass of particle
- \( M \) mass of shell
- \( r \) distance from particle to center of spherical shell
• **Gravitation inside the Earth**

A **uniform** shell of matter exerts **no** net gravitational force on a particle located **inside** it.

\[ \frac{r_1}{d_1} = \frac{r_2}{d_2}, \quad \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} \]

\[ \frac{F_1}{F_2} = \frac{A_1/d_1^2}{A_2/d_2^2} = 1 \]

\[ \vec{F}_1 + \vec{F}_2 = 0 \]
The gravitational force on the particle is the same as the force due to a sphere of radius $r$.

\[ F = \frac{Gm}{r^2} \times \text{Mass inside sphere of radius } r \]

\[ = \frac{Gm}{r^2} \times \left( \frac{4\pi r^3 \rho}{3} \right) = \frac{4\pi Gm \rho}{3} r = \frac{GmM}{R^3} r \]
• The gravitational force is conservative

• Gravitational potential energy:

\[ U(\infty) - U(r) = -W_{Fg,r \to \infty} = \frac{GmM}{r} \]

\[ U(r) = -\frac{GmM}{r} \]
Gravitational potential energy: system of particles

- For any pair of particles $(i, j)$

\[ U_{ij} = -\frac{G m_i m_j}{r_{ij}} \]

\[ U_{\text{total}} = -\sum_{i<j} \frac{G m_i m_j}{r_{ij}} \]

\[ U_{\text{total}} = -\left( \frac{G m_1 m_2}{r_{12}} + \frac{G m_2 m_3}{r_{23}} + \frac{G m_1 m_3}{r_{13}} \right) \]
Escape Speed

Initially: object on the surface of the Earth; kinetic and potential energy

\[ E_{mec} = K_0 + U_{grav} \]

Finally: object at \( \infty \); kinetic energy

\[ E_{mec} = K \geq 0 \]

\[
\frac{1}{2}mv_0^2 - \frac{GmM}{R} = \frac{1}{2}mv^2 \geq 0 \quad \Rightarrow \quad v_0 \geq \sqrt{\frac{2GM}{R}}
\]
• Planets and satellites: Kepler’s 1st law

How do objects – planets and satellites – move under gravitational force?

Light objects under the gravitational force of a very massive body assumed stationary (Sun or Earth).

\[ M_{\text{Sun}} \gg M_{\text{Planet}} \quad M_{\text{Earth}} \gg M_{\text{Satellite}} \]

COM of the system \( \simeq \) COM of central body
The Law of Orbits

All planets move in elliptical orbits with the sun at one focus.

**Ellipse:**
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a \geq b) \]

- \( a = \) (semi)major axis
- \( b = \) (semi)minor axis

**Excentricity:**
\[ e = \sqrt{1 - \frac{b^2}{a^2}} \]
\[ f = ae = d(C, F) = d(C, F') \]

- \( R_p = \) perihelion
- \( R_a = \) aphelion
The Law of Areas

A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet’s orbit in equal time intervals; that is, the rate $\frac{dA}{dt}$ at which it sweeps out area $A$ is constant.
The Law of Periods

The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

For a circular orbit \((a = b = r)\) Newton’s 2nd law

\[
\vec{F}_g = m\vec{a}_c \implies \frac{GmM}{r^2} = m\omega^2 r
\]

\[
T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{r^3}{GM}} \implies T^2 = \left(\frac{4\pi^2}{GM}\right) r^3
\]

For any elliptical orbit:

\[
T^2 = \left(\frac{4\pi^2}{GM}\right) a^3
\]
• **Satellites: orbits and energy**

- **Elliptical** orbit:
  
  \[ E = -\frac{GMm}{2a} \]

- **Circular** orbit:

  \[
  \frac{GMm}{r^2} = \frac{mv^2}{r} \\
  \downarrow \\
  E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}
  \]

  \[ E = -\frac{GMm}{2r} \]

- **Potential energy:**

  \[ U = -\frac{GMm}{r} \]
A space shuttle is initially in a circular orbit of radius $r$ about Earth. At point $P$ the pilot briefly fires a forward-pointing thruster to decrease the shuttle’s kinetic energy $K$ and mechanical energy $E$. Which of the dashed elliptical orbits shown in the figure will the shuttle then take?

$A$) inner orbit  
$B$) outer orbit
A space shuttle is initially in a circular orbit of radius $r$ about Earth. At point $P$ the pilot briefly fires a forward-pointing thruster to decrease the shuttles kinetic energy $K$ and mechanical energy $E$. Which of the dashed elliptical orbits shown in the figure will the shuttle then take?

$E = -\frac{GMm}{2a}$

$E \downarrow \Leftrightarrow a \downarrow$

$A)$ inner orbit

$B)$ outer orbit
In the same situation, is the orbital period $T$ of the shuttle (the time to return to $P$) then

A) greater than,  
B) less than, or  
C) the same as  
in the circular orbit?
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$A)$ greater than,

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in the circular orbit?

$T^2 \sim a^3$
Final Exam

• Chapter 7 – Kinetic Energy and work

• Chapter 8 – Potential energy and conservation

• Chapter 9 – COM and momentum. Systems of varying mass (Sect 9.9) included. Two dimensional collisions (Sect 9.8) included.

• Chapter 13 – Gravitation; Sections 13.1-13.5 and 13.7. Kepler’s 2nd and 3rd laws in Sect 13.6 are not included. Kepler’s first law in Section 13.6 and orbits and energy (Sect 13.7) are included.
6. A very narrow tunnel is drilled through a uniform spherical planet of radius $R$ and mass $M$, passing through the center of the planet. A small object is dropped with zero initial velocity from one end of the tunnel and falls toward the center. Neglecting friction and air drag, let $a$ be the acceleration of the object when it reaches a distance $2R/3$ from the center. Then the ratio

$$\frac{a}{GM/R^2}$$

is

a) $1/2$

b) $3/2$

c) $1$

d) $1/3$
e) \(2/3\)

Using the sphericall shell theorem, the radial component of the force on the object at distance \(r\) from the center is

\[
F_r = -\frac{4\pi}{3} \rho r^3 \frac{GM}{r^2}
\]

where

\[
\rho = \frac{M}{(4\pi/3)R^3}
\]

is the density of the planet. Therefore

\[
F_r = -\frac{GMmM}{R^3}r.
\]
The acceleration at distance $2R/3$ from the center is

$$a = \frac{2GM}{3R^2}.$$ 

7. In the previous problem, suppose the initial velocity of the object is zero. Then what is its speed when it reaches the center?

a) $\sqrt{GM/R}$

b) $\sqrt{GM/2R}$

c) $\sqrt{3GM/2R}$

d) $\sqrt{2GM/R}$

e) $\sqrt{2GM/3R}$
The force is the same as a spring force with constant

\[ k = \frac{GmM}{R^3}. \]

Therefore the work done by the gravitational force until the falling object reaches the center is

\[ W_g = \frac{1}{2} k R^2 = \frac{GmM}{2R} \]

Then

\[ \Delta K = \frac{mv^2}{2} = W_g \]

which yields

\[ v = \sqrt{GM/R}. \]
9. A space shuttle moves on a circular orbit of radius $r$ with constant speed $v$ around a planet of mass $M$. At some point the shuttle fires its engines and reduces its speed to $v/\sqrt{2}$. Find the semi-major axis of the resulting elliptic trajectory.

- a) $2r/3$
- b) $r/2$
- c) $3r/4$
- d) $3r/2$
- e) $4r/3$

The total mechanical energy of the shuttle on an
elliptic trajectory is

\[ E = -\frac{GmM}{2a} \]

where \( m \) is the mass of the shuttle and \( a \) the semi-major axis.

For the circular trajectory the gravitational force must be equal to the centripetal force:

\[ \frac{mv^2}{r} = \frac{GmM}{r^2}. \]

Therefore the kinetic energy on the circular trajectory is

\[ K = \frac{GmM}{2r}. \]

When the speed is reduced to \( v/\sqrt{2} \), the kinetic energy is reduced by half. At the same time the gravitational
potential energy stays the same:

\[ U = -\frac{GmM}{r}. \]

Therefore for the elliptical trajectory we have

\[ \frac{GmM}{4r} - \frac{GmM}{r} = -\frac{GmM}{2a}. \]

This yields

\[ a = \frac{2r}{3}. \]
12. A ball of mass $m = 0.1\text{kg}$ is dropped from a height $h = 11\text{m}$ above the surface of a triangular block of mass $M = 1\text{kg}$ as shown below. The triangular block is at rest on a frictionless table and the kinetic energy is conserved during the collision. Suppose the ball moves horizontally immediately after it hits the block. Find the speed of the block after collision.

(a) $0.5\text{m/s}$
(b) 0.1m/s
(c) 1m/s
(d) 1.2m/s
(e) 1.4m/s

**Solution.** The velocity of the ball just before it hits the block is

\[ \vec{v}_0 = -\sqrt{2gh}\hat{j}. \]

Let \( \vec{v}_1 = v_1\hat{i} \) denote the velocity of the ball after collision, which is horizontal by assumption. Let \( \vec{u} = u_x\hat{i} \) denote the velocity of the block after collision. Since there is no external force along the \( x \)-axis, the \( x \)-component of the total momentum is conserved.
Therefore

\[ Mu_x + mv_1 = 0 \]

Moreover, since the kinetic energy is conserved during the collision, we have

\[ Mu_x^2 + mv_1^2 = mv_0^2 \]

The above equations imply

\[ u = \sqrt{\frac{2m^2}{M(m + M)}}gh = 1.4 \text{m/s} \]
13. A triangular plate of angle $\theta = 30^\circ$ is glued to the surface of a horizontal table. A ball moving with speed $v_0 = 10 \text{ m/s}$ hits the plate sideways as shown below. The picture shows the collision seen from above such that the $(x, y)$ plane is horizontal. There is no friction between the ball and the plate and no friction between the ball and the table. Suppose the ball slides along the face of the plate after collision. Find the final speed of the ball.

For this problem note that

$$\sin(30^\circ) = \frac{1}{2} \quad \text{and} \quad \cos(30^\circ) = \frac{\sqrt{3}}{2}.$$
Solution. Choose a coordinate system as follows.
The only force acting on the ball during the collision is the normal force $\vec{N}$ which is orthogonal to the surface. Therefore the parallel component of the velocity must be preserved during the collision. This yields

$$mv_{\parallel} = mv_0 \sin \theta$$

after the collision. Hence $v_{\parallel} = v_0 \sin \theta = 5\text{m/s}$. 
14. In problem 13 suppose the mass of the ball is \( m = (\sqrt{3}/10) \) kg. Compute the magnitude of the impulse of the normal force acting on the ball during the collision.

(a) 1 kg \cdot m/s

(b) 1.5 kg \cdot m/s

(c) 0.5 kg \cdot m/s

(d) 0.7 kg \cdot m/s

(e) 1.7 kg \cdot m/s

Solution. The impulse of the normal force acting on the ball is given by

\[ \vec{J}_N = \Delta \vec{p}_\perp. \]
Since the ball slides along the surface after collision,
\[ \Delta p_\perp = mv_0 \cos \theta = 1.5 \text{ kg} \cdot \text{m/s}. \]
Therefore
\[ J_N = 1.5 \text{ kg} \cdot \text{m/s}. \]
15. A spring of constant $k$ is compressed a distance $\Delta l$ m between two blocks of masses $m$ and $2m$ respectively, lying on a smooth horizontal surface. The spring is assumed to be ideal and massless.

\[ \text{The blocks are free to move off the spring once it reaches its underformed configuration. Let } v \text{ be the final speed of the block of mass } m \text{ after the spring is released. Then the ratio } \frac{v}{\Delta l} \text{ is} \]

\begin{align*}
(a) & \quad \sqrt{\frac{2k}{3m}} \\
(b) & \quad \sqrt{\frac{k}{m}}
\end{align*}
(c) $\sqrt{3k/4m}$

(d) $\sqrt{k/2m}$

(e) $\sqrt{4k/5m}$

Energy conservation:

$$k(\Delta l)^2/2 = mv^2/2 + (2m)v_2^2/2$$

Momentum conservation:

$$mv = 2mv_2 \Rightarrow v_2 = v/2$$

Hence

$$k(\Delta l)^2 = m(1 + 1/2)v^2 = 3mv^2/2.$$ 

Therefore

$$\frac{v}{\Delta l} = \sqrt{2k/3m}.$$
16. A ball moves on a smooth horizontal surface with constant initial velocity vector
\[ \vec{v}_i = 3\hat{i} - 2\hat{j} \]
The ball hits a wall parallel with the \( x \) axis and bounces back with velocity vector \( \vec{v}_f \) such that
\[ v_{f,x} = \frac{v_{i,x}}{2}. \]
During the collision the \( x \) and \( y \) components of the force acting on the ball are related by
\[ F_x = -\frac{F_y}{2}. \]
What is the \( y \) component of the velocity of the ball \( \vec{v}_f \) after collision?
(a) 2 m/s
(b) 0.5 m/s
(c) 0 m/s
(d) 1 m/s
(e) 3 m/s

Using Newton’s 2nd law we have

\[ m(v_{fx} - v_{ix}) = J_x \quad m(v_{fy} - v_{iy}) = J_y \]

where \( \vec{J} = \int \vec{F} \, dt \) is the impulse of the force acting on the ball during the collision. Since \( F_x = -F_x/2 \) one has \( J_x = -J_y/2 \) as well. Hence

\[ v_{fy} - v_{iy} = -2(v_{fx} - v_{ix}) = v_{ix} \]

\[ v_{fy} = v_{iy} + v_{ix} = 1 \text{ m/s}. \]
18. An open cart is rolling in the rain on a horizontal surface without friction. The cart moves in a straight line along the $x$-axis. The rain droplets fall vertically into the cart such that the mass of the cart increases at the rate

$$\frac{1}{m} \frac{dm}{dt} = 0.2 \ s^{-1}.$$

Suppose at some moment the $x$-component of the velocity of the cart is $v_x = 1.2 \ m/s$. What is the magnitude of the acceleration of the cart at the same time, measured in $m/s^2$?

(a) 0.24

(b) 0.42

(c) 0.18
(d) 1.2
(e) 0.86

Let $dm$ be the increase in the mass of the cart and the water in it for an infinitesimal time interval, $t$ to $t + dt$. Let $v_x$ be the horizontal component of the velocity of the cart at time $t$. Momentum conservation yields

$$mv_x = (m + dm)(v_x + dv_x)$$

where $dv_x$ is the infinitesimal variation of the horizontal velocity of the cart in the same time interval. This yields

$$(dm)v_x + mdv_x = 0,$$
hence
\[ a_x = \frac{dv_x}{dt} = -v_x \frac{1}{m} \frac{dm}{dt} = -0.24 \text{ m/s}^2. \]

21. A ball tied at one end of a massless cord of length \( l = 1\text{ m} \) is free to move on a frictionless horizontal surface. The other end of the cord is fixed at the origin \( O \) and the ball is initially at \( (x_0, y_0) = (0, 0.5\text{ m}) \) such that the cord is slack. Suppose the ball is given an initial velocity \( \vec{v}_0 = (2\text{ m/s})\hat{i} \) and it starts moving on a circular trajectory of radius \( l \) as soon as the cord is stretched taut. Find the speed of the ball on the circular trajectory.
(a) 1m/s
(b) 2m/s
(c) 0.5m/s
(d) 1.5m/s
(e) 2.5m/s

The picture below shows the configuration of the system when the cord is stretched taut. Let $\vec{v}_0\parallel$ and $\vec{v}_0\perp$ denote the components of $\vec{v}_0$ with respect to the
direction of the cord. Note that the tension in the cord acts only along the parallel direction, and there is no force acting perpendicular to the cord. Therefore the perpendicular component of the momentum is conserved. This yields

\[ mv = mv_0 \sin \theta. \]

From the right triangle we have \( \sin \theta = \frac{d}{l} = \frac{1}{2} \). Therefore

\[ v = \frac{v_0}{2} = 1 \text{m/s}. \]
22. Suppose the mass of the ball is $m = 0.1$ kg in problem 21. Compute the magnitude of the impulse of the tension force in the cord during the transition from linear to circular motion.

(a) $\sqrt{3}/5$ kg $\cdot$ m/s
(b) $\sqrt{3}/2$ kg $\cdot$ m/s
(c) $1/3$ kg $\cdot$ m/s
(d) $1/5$ kg $\cdot$ m/s
(e) $\sqrt{3}/10$ kg $\cdot$ m/s

The impulse of the tension force is given by

$$\vec{J} = \Delta \vec{p}_\parallel$$
since tension only acts along the cord. Therefore

\[ J = mv_0 \cos \theta = \frac{\sqrt{3}}{10} \text{ kg} \cdot \text{m/s}. \]

29. An empty spherical cavity is cut inside a uniform hollowed sphere of mass density \( \rho \) and radius \( R \) as shown below. The spherical cavity has radius \( R \) and is centered at distance \( R \) from the center. A pointlike particle of mass \( m \) is placed right at the center of the hollowed sphere.
Let $F_g$ denote the magnitude of the gravitational force acting on the particle. Then the ratio

$$\frac{F_g}{\pi G m \rho R}$$

is:

a) $3/4$

b) $3/2$

c) $1/2$

d) $1/3$
e) \(2/3\)

In the absence of the cavity the force acting on a particle at the center is 0. Therefore, by superposition,

\[
\vec{F}_g + \vec{F}_{\text{small sphere}} = 0
\]

where \(\vec{F}_{\text{small sphere}}\) is the gravitational force acting on a particle at the surface of a uniform spherical distribution of mass of radius \(R/2\). This yields

\[
F_g = \frac{4\pi m \rho G R}{3} \cdot \frac{2}{2} = \frac{2\pi m \rho G R}{3}
\]

30. A spring pulls a block of mass \(m\) along a frictionless horizontal table as shown below. The block
is subject to air drag of magnitude

\[ D = cv^2 \]

for some coefficient \( c \).

Suppose at some moment during the motion the spring is elongated, and the velocity and the acceleration of the block are parallel to the spring and point to the left. At the same time the magnitude of the
acceleration is

\[ a = \frac{2cv^2}{5m}. \]

Then the instantaneous power output of the spring at this moment is

a) \( \frac{2cv^3}{5} \)
b) \( \frac{7cv^3}{5} \)
c) \( \frac{cv^3}{5} \)
d) \( \frac{6cv^3}{5} \)
e) \( cv^3 \)

Newton’s second law:

\[ F_{\text{spring}} - cv^2 = ma \]

\[ F_{\text{spring}} = cv^2 + ma = \frac{7}{5}cv^2 \]
Power

\[ P = \vec{F} \cdot \vec{v} = \frac{7}{5}cv^3 \]

19. A compressed spring launches a block of mass \( m = 0.1\text{kg} \) up a circular ramp of radius \( R = 0.5\text{m} \) such that the top point \( P \) is a height \( R = 0.5\text{m} \) above the ground. The horizontal surface is frictionless, but there is nonzero kinetic friction between the block and the ramp. The spring constant is \( k = 100\text{N/m} \) and the spring is compressed a distance \( \Delta l = 0.2\text{m} \). Suppose the block reaches the point \( P \) with speed \( v = 4\text{m/s} \). What is the work done by kinetic friction during the motion?
The work done by friction equals the variation of the

(a) 0.93J  
(b) \(-0.71\)J  
(c) \(-0.49\)J  
(d) 1.22J  
(e) \(-1.18\)J
mechanical energy:

\[ W_{fk} = mgR + \frac{mv^2}{2} - \frac{k(\Delta l)^2}{2} = 0.49 + 0.8 - 2 = -0.71 \text{J} \]