Rutgers University
Department of Physics & Astronomy

Honors Analytical Physics I

Lecture 13
• **Final Exam**: Tuesday, December 19th, 4-7pm

• Last name A-L: SERC 117

• Last name M-Z: SERC 118
Collisions in closed isolated system

- kinetic energy conserved $\Rightarrow$ elastic collisions

- kinetic energy not conserved, transferred to other forms of energy such as thermal energy $\Rightarrow$ inelastic collisions
Completely inelastic collisions

The objects stick together after collision.

\[ \vec{P}_i = \vec{P}_f \]

\[ \vec{V} = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2} \]
• **Elastic collisions in 1D: both** linear momentum and kinetic energy are conserved.

\[ \vec{P}_i = \vec{P}_f, \quad K_i = K_f \]
\[ \begin{align*}
  v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\
  v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} 
\end{align*} \]
• **Collisions in 2D**

A glancing collision that conserves both momentum and kinetic energy.

• Linear momentum conserved:

\[ \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \]

• Stationary target:

\[ m_1 v_{1i} = m_1 v_{1f} \cos \theta_1 + m_2 v_{2f} \cos \theta_2 \]
\[ m_1 v_{1f} \sin \theta_1 = m_2 v_{2f} \sin \theta_2 \]

• **If elastic**, kinetic energy also conserved:

\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]
• Systems with varying mass

(a) accelerating rocket at time $t$ in inertial frame

(b) accelerating rocket at time $t + dt$ in the same frame

$v \rightarrow v + dv, \quad dv > 0 \quad M \rightarrow M + dM, \quad dM < 0$
Suppose the relative speed \( v_{rel} \) between the rocket and exhaust products is known.

How do we find the acceleration?
Rocket + exhaust products = isolated closed system

\[ \vec{P}_a = \vec{P}_b \quad P_{a,x} = P_{b,x} \]
The ejection of mass from the rocket’s rear increases the rocket’s speed.

\[ P_{a,x} = Mv \quad P_{b,x} = (M + dM)(v_x + dv_x) + (-dM)u_x \]

**Note:** \( u_x \) the \( x \)-component of the velocity of the exhaust products **relative to the inertial frame**

\[ v_x + dv_x = u_x + v_{rel} \]
The ejection of mass from the rocket's rear increases the rocket's speed.

\[ Mv = (M + dM)(v_x + dv_x) - (v_x + dv_x - v_{rel})dM \]

\[ M dv_x + v_{rel}dM = 0 \quad \Rightarrow \quad M \frac{dv_x}{dt} = -v_{rel} \frac{dM}{dt} \]
The 1st rocket equation

\[ Ma_x = -v_{rel} \frac{dM}{dt} = Rv_{rel} \]
The ejection of mass from the rocket's rear increases the rocket's speed.

\[ v_{f,x} - v_{i,x} = v_{rel} \ln \frac{M_i}{M_f} \]

**The 2nd rocket equation**
13. Gravitation

Newton’s law of gravitation

Every point particle attracts every other particle with a gravitational force

\[ F = G \frac{m_1 m_2}{r^2} \]

\[ G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 = 6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \]
Gravitational force on particle 1 – vector form:

\[ \vec{F}_{2\text{on}1} = G \frac{m_1 m_2}{r^2} \hat{r} \]

\( \hat{r} \) is the radial \textbf{unit} vector: \(|\hat{r}| = 1\).
• **Principle of Superposition**

Given $n$ interacting particles:

\[
\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1n}
\]

$\vec{F}_{1,\text{net}} =$ net gravitational force acting on particle 1

$\vec{F}_{1,i} =$ gravitational force on particle 1 from particle $i$
Example

We want the forces (pulls) on particle 1, not the forces on the other particles.

- Isolated system of particles far from other massive objects.
- What is the magnitude of the gravitational force on particle 1?

\[ \vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} \]
\[
\vec{F}_{12} = G \frac{m_1 m_2 \hat{j}}{a^2}
\]

\[
\vec{F}_{13} = -G \frac{m_1 m_3 \hat{i}}{4a^2}
\]

\[
\vec{F}_{1,\text{net}} = \frac{G m_1}{a^2} \left( m_2 \hat{j} - \frac{m_3}{4} \hat{i} \right)
\]

\[
|\vec{F}_{1,\text{net}}| = \frac{G m_1}{a^2} \sqrt{m_2^2 + \frac{m_3^2}{16}}
\]
- Gravitational force on a particle from an extended object:

\[
\vec{F} = \int d\vec{F} = \int -G \frac{mdM}{r^2} \hat{r}
\]
Shell theorem

A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell’s mass were concentrated at its center.

\[ F = G \frac{mM}{r^2} \]

- \( m \) mass of particle
- \( M \) mass of shell
- \( r \) distance from particle to center of spherical shell

(http://hyperphysics.phy-astr.gsu.edu/hbase/mechanics/sphshell.html)
Consequence:
The gravitational force between two uniform spherical distributions of mass is the same as if all the mass of each sphere were concentrated at its center.
• **Gravitation near Earth’s surface**

• Assume the Earth is exactly spherical.

• Assume uniform distribution of mass throughout the Earth.

• Neglect rotation effects.

Gravitational force on a particle near Earth’s surface:

\[ \vec{F}_g = -G \frac{mM}{r^2} \hat{r} \]

\[ a_g = \frac{GM}{r^2} \]

\( r = \) distance between particle and center of the Earth.
Note: the free fall acceleration decreases with $r$

\[
 a_g = \frac{GM}{r^2} = \frac{GM}{(R + h)^2}
\]

$h = \text{altitude} = \text{distance from particle to the surface of the Earth}$

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>$a_g$ (m/s²)</th>
<th>Altitude Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.83</td>
<td>Mean Earth surface</td>
</tr>
<tr>
<td>8.8</td>
<td>9.80</td>
<td>Mt. Everest</td>
</tr>
<tr>
<td>36.6</td>
<td>9.71</td>
<td>Highest crewed balloon</td>
</tr>
<tr>
<td>400</td>
<td>8.70</td>
<td>Space shuttle orbit</td>
</tr>
<tr>
<td>35 700</td>
<td>0.225</td>
<td>Communications satellite</td>
</tr>
</tbody>
</table>
Rotation effects

Two forces act on this crate.

The normal force is upward.

The net force is toward the center. So, the crate's acceleration is too.

\[ m\vec{a}_c = m\vec{a}_g + \vec{F}_N \]

\[ F_N = ma_g - m\omega^2 R \]

\[ g = a_g - \omega^2 R \]
• **Gravitation inside the Earth**

A *uniform* shell of matter exerts *no* net gravitational force on a particle located *inside* it.

\[
\vec{F}_1 + \vec{F}_2 = 0
\]
Suppose a very narrow tunnel is dug through the Earth along the North-South axis.

What is the gravitational force on a particle of mass \( m \) at distance \( r < R \) from the center?

Any thin spherical shell of matter of radius \( r_{\text{shell}} > r \) does not yield any net force.

Only thin shells of radius \( r_{\text{shell}} < r \) give a nonzero force.
The gravitational force on the particle is the same as the force due to a sphere of radius \( r \).

Assume spherical shape and uniform mass distribution. Mass density

\[
\rho = \frac{M}{4\pi R^3/3} = \frac{3M}{4\pi R^3}
\]

\[
F = \frac{Gm}{r^2} \times \text{Mass inside sphere of radius } r
\]

\[
= \frac{Gm}{r^2} \times \left( \frac{4\pi r^3 \rho}{3} \right) = \frac{4\pi Gm \rho}{3}r = \frac{GmM}{R^3}r
\]
Gravitational potential energy

Work done by gravitational force: suppose a baseball moves upward from a distance \( r_1 \) to a distance \( r_2 > r_1 \) in the gravitational field of the Earth.

What is the work done by gravitational force?
\[ W_{Fg} = \int_{r_1}^{r_2} \vec{F}_g \cdot d\vec{r} \]

\[ = -GmM \int_{r_1}^{r_2} \frac{dr}{r_2} \]

\[ = GmM \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \]

Does it depend on the path?
$W_{Fg}$ does not depend on the path.

Contribution of circular arcs is 0:

$$\vec{F}_g \cdot d\vec{r} = 0$$

Contribution of radial arcs:

$$\vec{F}_g \cdot d\vec{r} = \frac{-GmM}{r^2} dr$$

$W_{Fg,\text{curved path}} = W_{Fg,\text{radial path}}$
The gravitational force is conservative

Gravitational potential energy:

\[ U(\infty) - U(r) = -W_{Fg,r \rightarrow \infty} = \frac{GmM}{r} \]

\[ U(r) = -\frac{GmM}{r} \]
Gravitational potential energy: system of particles

- For any pair of particles $(i, j)$

$$U_{ij} = -\frac{Gm_i m_j}{r_{ij}}$$

$$U_{\text{total}} = -\sum_{i<j} \frac{Gm_i m_j}{r_{ij}}$$

$$U_{\text{total}} = -\left(\frac{Gm_1 m_2}{r_{12}} + \frac{Gm_2 m_3}{r_{23}} + \frac{Gm_1 m_3}{r_{13}}\right)$$
• **Escape speed**

How fast should a projectile be launched such that it escapes Earth’s gravitational field?

How fast should it be launched such that it does not fall back on Earth?