• **Final Exam:**

• Friday December 20, 4:00-7:00pm

• Physics Lecture Hall

• Energy, Center of Mass, Momentum, Gravity

• Ch. 7,8,9 and 13.1-13.8
• **Midterm II:**

\[
\begin{array}{cccccccccccc}
15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\
3 & 8 & 9 & 14 & 15 & 13 & 13 & 12 & 10 & 5 & 2 & 1 & 1 \\
\end{array}
\]

Average:

\[
\frac{10.07}{15} = 66.67\%
\]
9. Center of Mass. Linear Momentum I

The center of mass of a system of particles is the point that moves as though:

1) all mass of the system were concentrated there and

2) all external forces applied there.

\[
\vec{r}_{\text{com}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M} \sum_{i=1}^{N} m_i\vec{r}_i
\]
• Newton's 2nd law for a system of particles

\[ \vec{F}_{\text{net}} = M \vec{a}_{\text{com}} \]

\[ \vec{v}_{\text{com}} = \frac{1}{M} (m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots + m_N \vec{v}_N) \]

\[ \vec{a}_{\text{com}} = \frac{1}{M} (m_1 \vec{a}_1 + m_2 \vec{a}_2 + \cdots + m_N \vec{a}_N) \]
• **Linear momentum of a system of particles**

The linear momentum of a system of particles is equal to the product of the total mass \( M \) of the system and the velocity of the center of mass.

\[
\vec{P} = M\vec{v}_{\text{com}}
\]

\[
\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i\vec{v}_i = M\vec{v}_{\text{com}}
\]
The rate of change of the momentum is equal to the net external force acting on the system:

\[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \]
Conservation of linear momentum

A system is **closed** if no particles leave or enter the system.

A system is **isolated** if no external forces act on the system.

**Isolated closed system:**

\[
\frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} = 0 \implies \vec{P} \text{ conserved}
\]

If no net external force acts on a closed system of particles, the total linear momentum \( \vec{P} \) of the system cannot change.
Closed system:

\[(F_{\text{ext}})_x = 0 \Rightarrow \frac{dP_x}{dt} = 0 \Rightarrow P_x \text{ conserved}\]

\[(F_{\text{ext}})_y = 0 \Rightarrow \frac{dP_y}{dt} = 0 \Rightarrow P_y \text{ conserved}\]

\[(F_{\text{ext}})_z = 0 \Rightarrow \frac{dP_z}{dt} = 0 \Rightarrow P_z \text{ conserved}\]

If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.
• Momentum and kinetic energy in collisions

Collisions in closed isolated system

• kinetic energy conserved $\Rightarrow$ elastic collisions

• kinetic energy not conserved, transferred to other forms of energy such as thermal energy $\Rightarrow$ inelastic collisions
• **Completely inelastic collisions in 1D**

**Completely inelastic:** the objects stick together after collision.
$\vec{P}_i = \vec{P}_f$

$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$
In a completely inelastic collision, the bodies stick together.

\[ \vec{P}_i = \vec{P}_f \]

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{V} \]

\[ \vec{V} = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2} \]

\[ V_x = \frac{m_1 v_{1i,x} + m_2 v_{2i,x}}{m_1 + m_2} \]
The com of the two bodies is between them and moves at a constant velocity.

Here is the incoming projectile.

Here is the stationary target.

Collision!

The com moves at the same velocity even after the bodies stick together.

$$\vec{V} = \vec{v}_{\text{com}}$$
An object at rest explodes into two pieces of unequal mass. One piece flies west at a speed $v$ and the second flies east at a speed $3v$. What is the velocity of the center of mass?

A) 0  
B) $2v$ west  
C) $2v$ east  
D) cannot be determined
An object at rest explodes into two pieces of unequal mass. One piece flies west at a speed $v$ and the second flies east at a speed $3v$. What is the velocity of the center of mass?

$A$) 0  

$B$) 2$v$ west  

$C$) 2$v$ east  

$D$) cannot be determined  

$\vec{P}_i = \vec{P}_f = M\vec{v}_{com} = 0$
**Example: ballistic pendulum**

- A large block of wood of mass \( M = 5.4 \text{ kg} \) suspended from two long cords.
- A bullet of mass \( m = 9.5 \text{ g} \) fired into the block.
- The system block + bullet swings upward a vertical distance \( h = 6.3 \text{ cm} \).
- \( v_{\text{bullet}} \) ?
**Step 1:**
- completely inelastic collision
- linear momentum conserved
- kinetic energy not conserved

\[ mv = (m + M)V \]
Step 2:  
- upward swing  
- linear momentum not conserved  
- mechanical energy conserved

\[ E_{\text{mec}} = K + U = \text{constant} \]

\[ \frac{1}{2}(M+m)V^2 = (M+m)gh \]

\[ V = \sqrt{2gh} \]

\[ v = \frac{m + M}{m} \sqrt{2gh} \]
**Example:** generic inelastic collision

A bullet of mass $m = 10\, \text{g}$ moving directly upward at $v = 1000\, \text{m/s}$ strikes and passes through the center of mass of a $M = 5.0\, \text{kg}$ block initially at rest.

The bullet emerges from the block moving directly upward at $v_1 = 400\, \text{m/s}$.

To what maximum height does the block then rise above its initial position?
Step 1:  • inelastic collision
  • linear momentum conserved
  • kinetic energy not conserved

\[ \vec{P}_i = \vec{P}_f \]

\[ mv = mv_1 + Mv_2 \]

\[ v_2 = \frac{m}{M}(v - v_1) \]
Step 2:  
- upward motion  
- linear momentum not conserved  
- mechanical energy conserved  
- assume the block does not move much during the collision

\[ \frac{1}{2} M v_2^2 = M g h \]

\[ h = \frac{v_2^2}{2g} = \frac{m^2}{2M^2 g} (v - v_1)^2 \]
• **Elastic collisions in 1D: both** linear momentum and kinetic energy are conserved
• **Generic setup – stationary target**

  ![Generic setup diagram](https://via.placeholder.com/150)

  Here is the generic setup for an elastic collision with a stationary target.

  Before \( m_1 \) projectile \( v_{1i} \) \( v_{2i} = 0 \) \( x \)

  After \( m_1 \) projectile \( v_{1f} \) \( v_{2f} \) \( x \)

  - **The linear momentum of the systems is conserved:**
    \[
    \vec{P}_i = \vec{P}_f
    \]

  - **The total kinetic energy of the系统 is conserved:**
    \[
    K_i = K_f
    \]

  **Note:** the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.
Here is the generic setup for an elastic collision with a stationary target.

\[ m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \]

\[ \frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]

\[
\begin{align*}
 m_1 (v_{1i} - v_{1f}) &= m_2 v_{2f} \\
 m_1 (v_{1i} - v_{1f}) (v_{1i} + v_{1f}) &= m_2 v_{2f}^2 \\
 \Downarrow
\end{align*}
\]

\[
\begin{align*}
 v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \\
 v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i}
\end{align*}
\]
1D elastic collision – stationary target

\[ v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} \]

Note:

- \( v_{2f} > 0 \)
- \( v_{1f} > 0 \) if \( m_1 > m_2 \); \( v_{1f} < 0 \) if \( m_1 < m_2 \)
- \( v_{1f} = 0, \ v_{2f} = v_{1i} \) if \( m_1 = m_2 \) (identical particles)
• **Generic setup – moving target**

Here is the generic setup for an elastic collision with a moving target.

\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]

\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]
\[ m_1 v_1 i + m_2 v_2 i = m_1 v_1 f + m_2 v_2 f \]
\[ \downarrow \]
\[ m_1 (v_1 i - v_1 f) = m_2 (v_2 f - v_2 i) \]

\[ \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1^2 f + \frac{1}{2} m_2 v_2^2 f \]
\[ \downarrow \]
\[ m_1 (v_1 i - v_1 f) (v_1 i + v_1 f) = m_2 (v_2 f - v_2 i) (v_2 f + v_2 i) \]
\[ \downarrow \]
\[ v_1 i + v_1 f = v_2 f + v_2 i \]
1D elastic collision – moving target

\[ v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \]

\[ v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \]
Example: two pendulums

- How high will the 1st ball recoil after collision?
- Which way will it swing?
- How high will the 2nd ball swing after collision?
• **Step 1:**

\[
mgh_1 = \frac{1}{2}mv_{1i}^2
\]

\[
v_{1i} = \sqrt{2gh_1}
\]

• **Step 2:** collision

\[
v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}v_{1i}
\]

\[
v_{2f} = \frac{2m_1}{m_1 + m_2}v_{1i}
\]

• **Step 3:**

\[
m_1gh_{1f} = \frac{1}{2}m_1v_{1f}^2
\]

\[
m_2gh_{2f} = \frac{1}{2}m_2v_{2f}^2
\]
• **Collisions in 2D**

A glancing collision that conserves both momentum and kinetic energy.

- **Linear momentum conserved:**
  \[
  \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}
  \]

- **Stationary target:**
  \[
  m_1v_{1i} = m_1v_{1f}\cos\theta_1 + m_2v_{2f}\cos\theta_2
  
  m_1v_{1f}\sin\theta_1 = m_2v_{2f}\sin\theta_2
  \]

- **If elastic**, kinetic energy also conserved:
  \[
  \frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2
  \]
• **Systems with varying mass**

(a) accelerating rocket at time $t$ in inertial frame

(b) accelerating rocket at time $t + dt$ in the same frame

$v \rightarrow v + dv, \quad dv > 0 \quad M \rightarrow M + dM, \quad dM < 0$
Suppose the relative speed $v_{rel}$ between the rocket and exhaust products is known.

How do we find the acceleration?
Rocket + exhaust products = isolated closed system

\[ \downarrow \]

Linear momentum conserved

\[ \vec{P}_a = \vec{P}_b \quad P_{a,x} = P_{b,x} \]
The ejection of mass from the rocket's rear increases the rocket's speed.

\[ P_{a,x} = Mv \quad P_{b,x} = (M + dM)(v_x + dv_x) + (-dM)u_x \]

**Note:** \( u_x \) the \( x \)-component of the velocity of the exhaust products relative to the inertial frame

\[ v_x + dv_x = u_x + v_{rel} \]
\[ M\, v = (M + dM)(v_x + dv_x) - (v_x + dv_x - v_{rel})\, dM \]

\[ M\, dv_x + v_{rel}\, dM = 0 \quad \Rightarrow \quad M\, \frac{dv_x}{dt} = -v_{rel}\, \frac{dM}{dt} \]
Ma_x = \frac{-v_{rel} \, dM}{dt} = Rv_{rel}

The 1st rocket equation

The ejection of mass from the rocket’s rear increases the rocket’s speed.
The ejection of mass from the rocket's rear increases the rocket's speed.

\[ v_{f,x} - v_{i,x} = v_{rel} \ln \frac{M_i}{M_f} \]

The 2nd rocket equation
Rain falls vertically into an open cart rolling horizontally. What happens to the momentum, speed and kinetic energy?

\[ \begin{array}{ccc}
A) & p & v & K \\
 & \text{same} & \text{same} & \text{same} \\
B) & \text{increases} & \text{same} & \text{increases} \\
C) & \text{increases} & \text{increases} & \text{increases} \\
D) & \text{same} & \text{decreases} & \text{same} \\
E) & \text{same} & \text{decreases} & \text{decreases} \\
\end{array} \]
Rain falls vertically into an open cart rolling horizontally. What happens to the momentum, speed and kinetic energy?

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>v</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>same</td>
<td>same</td>
<td>same</td>
</tr>
<tr>
<td>B</td>
<td>increases</td>
<td>same</td>
<td>increases</td>
</tr>
<tr>
<td>C</td>
<td>increases</td>
<td>increases</td>
<td>increases</td>
</tr>
<tr>
<td>D</td>
<td>same</td>
<td>decreases</td>
<td>same</td>
</tr>
<tr>
<td>E</td>
<td>same</td>
<td>decreases</td>
<td>decreases</td>
</tr>
</tbody>
</table>
13. Gravitation

Newton’s law of gravitation

Every point particle attracts every other particle with a gravitational force

\[ F = G \frac{m_1 m_2}{r^2} \]

\[ G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 = 6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \]
Gravitational force on particle 1 – vector form:

\[
\vec{F}_{2\ \text{on} \ 1} = G\frac{m_1 m_2}{r^2} \hat{r}
\]

\(\hat{r}\) is the radial \textbf{unit} vector: \(|\hat{r}| = 1\).
Principle of Superposition

Given $n$ interacting particles:

\[
\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1n}
\]

$\vec{F}_{1,\text{net}}$ = net gravitational force acting on particle 1

$\vec{F}_{1,i}$ = gravitational force on particle 1 from particle $i$
Example

- Isolated system of particles far from other massive objects.
- What is the magnitude of the gravitational force on particle 1?

\[ \vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} \]
\[ \vec{F}_{12} = G \frac{m_1 m_2 \hat{j}}{a^2} \]

\[ \vec{F}_{13} = -G \frac{m_1 m_3 \hat{i}}{4a^2} \]

\[ \vec{F}_{1,\text{net}} = \frac{G m_1}{a^2} \left( m_2 \hat{j} - m_3 \frac{\hat{i}}{4} \right) \]

\[ |\vec{F}_{1,\text{net}}| = \frac{G m_1}{a^2} \sqrt{m_2^2 + \frac{m_3^2}{16}} \]
- Gravitational force on a particle from an extended object:

\[
\vec{F} = \int d\vec{F} = \int -G \frac{mdM}{r^2} \hat{r}
\]
**Shell theorem**

A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell’s mass were concentrated at its center.

\[ F = G \frac{mM}{r^2} \]

- \( m \) mass of particle
- \( M \) mass of shell
- \( r \) distance from particle to center of spherical shell

(http://hyperphysics.phy-astr.gsu.edu/hbase/mechanics/sphshell.html)
Consequence:
The gravitational force between two uniform spherical distributions of mass is the same as if all the mass of each sphere were concentrated at its center.
• **Gravitation near Earth’s surface**

  • Assume the Earth is exactly spherical.
  
  • Assume uniform distribution of mass throughout the Earth.
  
  • Neglect rotation effects.

Gravitational force on a particle near Earth’s surface:

\[
\vec{F}_g = -G \frac{mM}{r^2} \hat{r} \quad \Rightarrow \quad a_g = \frac{GM}{r^2}
\]

\( r = \text{distance between particle and center of the Earth} \)
Note: the free fall acceleration decreases with \( r \)

\[
a_g = \frac{GM}{r^2} = \frac{GM}{(R + h)^2}
\]

\( h \) = altitude = distance from particle to the surface of the Earth

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>( a_g ) (m/s(^2))</th>
<th>Altitude Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.83</td>
<td>Mean Earth surface</td>
</tr>
<tr>
<td>8.8</td>
<td>9.80</td>
<td>Mt. Everest</td>
</tr>
<tr>
<td>36.6</td>
<td>9.71</td>
<td>Highest crewed balloon</td>
</tr>
<tr>
<td>400</td>
<td>8.70</td>
<td>Space shuttle orbit</td>
</tr>
<tr>
<td>35700</td>
<td>0.225</td>
<td>Communications satellite</td>
</tr>
</tbody>
</table>
Rotation effects

Two forces act on this crate.

The normal force is upward.

The gravitational force is downward.

The net force is toward the center. So, the crate’s acceleration is too.

\[ m\ddot{a}_c = m\ddot{a}_g + \vec{F}_N \]

\[ F_N = ma_g - mv^2/R \]

\[ g = a_g - v^2/R \]
i-Clicker

The gravitational acceleration on the surface of the Earth is $g$ (neglecting rotation.) What will it be on the surface of a planet that has half the mass of the Earth and half its radius?

A) $g/4$
B) $g/2$
C) $g$
D) $2g$
The gravitational acceleration on the surface of the Earth is $g$ (neglecting rotation.) What will it be on the surface of a planet that has half the mass of the Earth and half its radius?

A) $g/4$

B) $g/2$

C) $g$

D) $2g$

\[
g = G \frac{M}{R^2} \Rightarrow g'/g = \frac{M'R^2}{M(R')^2} = \frac{4}{2} = 2
\]