• Midterm II: Sunday Nov 17, 11:30pm, PHL
9. Center of Mass. Linear Momentum I

The center of mass of a system of particles is the point that moves as though:

(1) all mass of the system were concentrated there and
(2) all external forces applied there.

The center of mass of the baseball bat follows a parabolic path, but all other points follow more complicated paths.
The center of mass: a system of particles

\[ x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M} \sum_{i=1}^{N} m_i x_i \]

Note: in general, it does not have to be one of the particles in the system.

Discrete linear distribution of particles
Three dimensions: discrete distribution of particles

\[ \vec{r}_{\text{com}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_N \vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i \]

\[ x_{\text{com}} = \frac{1}{M} \sum_{i=1}^{N} m_i x_i \]

\[ y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{N} m_i y_i \]

\[ z_{\text{com}} = \frac{1}{M} \sum_{i=1}^{N} m_i z_i \]
• **Solid body:** continuous mass distribution

\[ dm = \rho dv \]

\[ \mathbf{r}_{\text{com}} = x_{\text{com}} \mathbf{i} + y_{\text{com}} \mathbf{j} + z_{\text{com}} \mathbf{k} \]

\[ x_{\text{com}} = \frac{1}{M} \int x \, dm \quad y_{\text{com}} = \frac{1}{M} \int y \, dm \quad z_{\text{com}} = \frac{1}{M} \int z \, dm \]

\[ M = \int dm \quad \text{total mass of the object} \]
Mass density:

\[ \rho = \frac{dm}{dv} \quad dv = dxdydz \quad \text{volume element} \]

**Uniform** mass distribution:

\[ \rho = \text{constant as function of } \vec{r} \]

\[ \downarrow \]

\[ x_{\text{com}} = \frac{1}{V} \int x \, dv \quad y_{\text{com}} = \frac{1}{V} \int y \, dv \quad z_{\text{com}} = \frac{1}{V} \int z \, dv \]

\[ V = \int \, dv = \int \, dxdydz \quad \text{total volume} \]
• **Symmetry:** if the object has a symmetry axis and $\rho = \text{constant}$ then its com must lie on that axis.
Example

Consider an L shape made up of 4 blocks each a square with sides of length \( d \).

- Assume all blocks are uniform and have the same mass \( M \).
- Treat this as a 2-dimensional problem – ignore width in the \( z \)-direction.

\[
x_{\text{com}} = \frac{3M(d/2) + M(3d/2)}{4M} = \frac{3}{4}d
\]
i-Clicker

Consider an L shape made up of 4 blocks each a square with sides of length $d$.

- Assume all blocks are uniform and have the same mass $M$
- Treat this as a 2-dimensional problem - ignore width in the $z$-direction.

What is $y_{\text{com}}$?

$A) \ d \quad B) \ 7d/8 \quad C) \ 5d/4 \quad D) \ 3d/4 \quad E) \ d/2$
Consider an L shape made up of 4 blocks each a square with sides of length d.

- Assume all blocks have the same mass M
- Treat this as a 2-dimensional problem ignore width in the z-direction.

What is $y_{com}$?

A) $d$ B) $7d/8$ C) $5d/4$ D) $3d/4$ E) $d/2$
\[ y_{\text{com}} = \frac{2M(d/2) + M(3d/2) + M(5d/2)}{4M} = \frac{5}{4}d \]
• **Example:**

Consider the two uniform squares shown each with sides of length $d$.

- Assume $m_{\text{blue}} = 2 \text{ kg}$, $m_{\text{orange}} = 1 \text{ kg}$
- Treat this as a 2-dimensional problem.

$$x_{\text{com}} = \frac{m_{\text{blue}} x_{\text{blue}} + m_{\text{orange}} x_{\text{orange}}}{m_{\text{blue}} + m_{\text{orange}}} = -\frac{d}{6}$$
i-Clicker

Consider the two squares shown each with sides of length d.

- Assume \( m_{\text{blue}} = 2 \text{ kg} \), \( m_{\text{orange}} = 1 \text{ kg} \)
- Treat this as a 2-dimensional problem. What is \( y_{\text{com}} \) ?

\[\begin{align*}
A) & \quad d \\
B) & \quad d/4 \\
C) & \quad 0 \\
D) & \quad 3d/2 \\
E) & \quad d/2
\end{align*}\]
Answer

Consider the two squares shown each with sides of length d.

- Assume \( m_{\text{blue}} = 2 \text{ kg} \), \( m_{\text{orange}} = 1 \text{ kg} \)
- Treat this as a 2-dimensional problem. What is \( y_{\text{com}} \)?

\[ A) \ d \quad B) \ d/4 \quad C) \ 0 \quad D) \ 3d/2 \quad E) \ d/2 \]
\[ y_{\text{com}} = \frac{m_{\text{blue}} y_{\text{blue}} + m_{\text{orange}} y_{\text{orange}}}{m_{\text{blue}} + m_{\text{orange}}} = \frac{d}{2} \]
A metal plate $P$ is has the shape of a disk of radius $R$ with a hole of radius $R$. Assuming uniform density, $\rho = \text{constant}$, find $x_{\text{com}}$.

Treat the problem as two-dimensional ignoring thickness in the $z$-direction.
\( P \): holed disk

\( D \): whole big disk

\( H \): the small disk removed from \( D \)

\[ P = D - H \]

\[ x_{\text{com}}^{P} = \frac{\int_{P} x \, dx \, dy}{\int_{P} dx \, dy} = \frac{X_{P}}{A_{P}} \]

\[ X_{P} = \int_{D} x \, dx \, dy - \int_{H} x \, dx \, dy = X_{D} - X_{H} \]

\[ A_{P} = \int_{D} dx \, dy - \int_{H} dx \, dy = A_{D} - A_{H} \]
\begin{align*}
x_{\text{com}}^D &= 0 \quad x_{\text{com}}^H = -R \\
x_{\text{com}}^D &= \frac{X_D}{A_D} \quad x_{\text{com}}^H = \frac{X_H}{A_H}
\end{align*}

\begin{align*}
X_D &= 0, \quad X_H = -RA_H \\
x_{\text{com}}^P &= \frac{X_D - X_H}{A_D - A_H} = R \frac{A_H}{A_D - A_H} = \frac{R}{3}
\end{align*}
• Newton’s 2nd law for a system of particles

\[ \vec{F}_{\text{net}} = M \vec{a}_{\text{com}} \]

- \( \vec{F}_{\text{net}} \) is the net external force acting on the system. **Internal** forces are not included.

- \( M \) is the total mass of the system. If \( M \) is constant, the system is said to be **closed**.

- \( \vec{a}_{\text{com}} \) is the acceleration of the center of mass of the system.

\[
F_{\text{net},x} = M a_{\text{com},x} \\
F_{\text{net},y} = M a_{\text{com},y} \\
F_{\text{net},z} = M a_{\text{com},z}
\]
The internal forces of the explosion cannot change the path of the comet.

**Fig. 9-5** A fireworks rocket explodes in flight. In the absence of air drag, the center of mass of the fragments would continue to follow the original parabolic path, until fragments began to hit the ground.
Derivation

\[ M \vec{r}_{\text{com}} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_N \vec{r}_N \]

\[ \Downarrow \quad d/dt \]

\[ M \vec{v}_{\text{com}} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots + m_N \vec{v}_N \]

\[ \Downarrow \quad d/dt \]

\[ M \vec{a}_{\text{com}} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + \cdots + m_N \vec{a}_N \]

\[ M \vec{a}_{\text{com}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N \]
\[
M \vec{a}_{\text{com}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N
\]

\(\vec{F}_i\) = the sum of all forces acting on particle \(i\), including internal ones.

\[
\vec{F}_i = \vec{F}_{i,\text{ext}} + \vec{F}_{i,\text{int}}
\]

**Newton’s 3rd law:**

\[
\sum_{i=1}^{N} \vec{F}_{i,\text{int}} = 0
\]

\[
M \vec{a}_{\text{com}} = \vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}} + \cdots + \vec{F}_{N,\text{ext}}
\]
The com of the system will move as if all the mass were there and the net force acted there.

\[
M \vec{a}_{com} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3
\]

\[
M = m_1 + m_2 + m_3
\]
Problem 12. Ch. 9. Two skaters, one with mass 65 kg and the other with mass 40 kg, stand on an ice rink holding a pole of length 10 m and negligible mass. Starting from the ends of the pole, the skaters pull themselves along the pole until they meet. How far does the 40 kg skater move? Neglect friction.

$$\vec{F}_{\text{ext}} = 0 \Rightarrow \vec{a}_{\text{com}} = 0 \text{ and } \vec{v}_{\text{com}} = 0.$$ Hence the two skaters meet at the COM, which does not move.

Initially: \[ x_1 = 0 \quad x_2 = d \]

\[
x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_2 d}{m_1 + m_2} = \Delta x_1
\]
Suppose the kinetic friction coefficient between the skates and the ice is $\mu_k$. What is the acceleration of the center of mass?

**External force:**

\[
\vec{F}_{\text{ext}} = \vec{f}_{k1} + \vec{f}_{k2} \\
= \mu_k (m_2 - m_1) g \hat{i} \\
a_{\text{com},x} = \frac{m_2 - m_1}{m_1 + m_2} \mu_k g
\]
• Linear momentum

How can we predict the outcome of a collision?

Assuming energy is conserved

\[ K_1 + K_2 = K_1' + K_2' \]

Not sufficient!

Energy is a **scalar** quantity.

No direction!
Linear momentum of a particle

\[ \vec{p} = m \vec{v} \]

Momentum \( \vec{p} \) is a vector quantity; a particle’s momentum has the same direction as its velocity \( \vec{v} \).
The rate of change of the momentum is equal to the net force acting on the particle:

\[
\frac{d\vec{p}}{dt} = \vec{F}_{\text{net}}
\]

\[
\frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}_{\text{net}}
\]
• **Linear momentum of a system of particles**

The linear momentum of a system of particles is equal to the product of the total mass $M$ of the system and the velocity of the center of mass.

$$\vec{P} = M\vec{v}_{\text{com}}$$

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i = M\vec{v}_{\text{com}}$$
The rate of change of the momentum is equal to the net external force acting on the system:

\[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \]
Collision and Impulse

The figure depicts the collision at one instant. The ball experiences a force $F(t)$ that varies during the collision and changes the linear momentum of the ball.

Fig. 9-8  Force $\vec{F}(t)$ acts on a bat as the ball and a bat collide.

The collision of a ball with a bat collapses part of the ball. (Photo by Harold E. Edgerton. ©The Harold and Esther Edgerton Family Trust, courtesy of Palm Press, Inc.)
The change in linear momentum is related to the force by Newton’s second law:

\[ \vec{F} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \Delta \vec{p} = \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F} dt \]

- **Impulse:**

\[ \vec{J} = \int_{t_i}^{t_f} \vec{F} dt \]
• The magnitude of $\vec{J}$ equals the area under the curve $F(t)$.

• Average force

$$\vec{F}_{\text{average}} = \frac{\vec{J}}{\Delta t}$$

• Newton’s 3rd law:

$$\vec{F}_{\text{ball}}(t) + \vec{F}_{\text{bat}}(t) = 0$$

at all times. Hence:

$$\vec{J}_{\text{bat}} = -\vec{J}_{\text{ball}}$$

$$|\vec{J}_{\text{bat}}| = |\vec{J}_{\text{ball}}|$$
Consider two ways to slow a toy car from $15\text{km/h}$ to a complete stop.

(i) The car slams into a wall and stops.

(ii) The car gradually plows into a tube of gelatine and comes to a gradual halt.

In which case is the magnitude of the impulse bigger?

A) Case (i)  B) Case (ii)

C) They are equal.  D) Cannot be decided.
**Answer**

Consider two ways to slow a toy car from $15 \text{km/h}$ to a complete stop.

(i) The car slams into a wall and stops.

(ii) The car gradually plows into a tube of gelating and comes to a gradual halt.

In which case is the magnitude of the impulse bigger?

A) Case (i)  
B) Case (ii)  
C) They are equal.  
D) Cannot be decided.

\[ \vec{J} = \Delta \vec{p}, \text{ the same in both cases.} \]
The graphs below encode the time dependence of the force magnitude for a body involved in a collision. How are the impulse magnitudes ordered?

- **A)** $J_a > J_b > J_c$
- **B)** $J_a < J_b < J_c$
- **C)** $J_a = J_b > J_c$
- **D)** $J_a = J_b = J_c$
The graphs below encode the time dependence of the force magnitude for a body involved in a collision. How are the impulse magnitudes ordered?

A) $J_a > J_b > J_c$  
B) $J_a < J_b < J_c$  
C) $J_a > J_c < J_b$  
D) $J_a = J_b = J_c$
• **Conservation of linear momentum**

A system is **closed** if no particles leave or enter the system.

A system is **isolated** if no external forces act on the system.

**Isolated closed system:**

\[
\frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} = 0 \implies \vec{P} \text{ conserved}
\]

If no net external force acts on a closed system of particles, the total linear momentum \( P \) of the system cannot change.
Closed system:

\[ (F_{\text{ext}})_x = 0 \Rightarrow \frac{dP_x}{dt} = 0 \Rightarrow P_x \text{ conserved} \]

\[ (F_{\text{ext}})_y = 0 \Rightarrow \frac{dP_y}{dt} = 0 \Rightarrow P_y \text{ conserved} \]

\[ (F_{\text{ext}})_z = 0 \Rightarrow \frac{dP_z}{dt} = 0 \Rightarrow P_z \text{ conserved} \]

If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.
i-Clicker

A particle constrained to move on a horizontal smooth surface is subject to five forces as shown below. Which components of its momentum are conserved?

A) $p_x, p_y$ are conserved

B) $p_x$ conserved, $p_y$ not conserved

C) $p_x$ not conserved, $p_y$ conserved

D) $p_x, p_y$ not conserved.
A particle constrained to move on a horizontal smooth surface is subject to five forces as shown below. Which components of its momentum are conserved?

\[ F_x = \frac{dp_x}{dt} = 0 \]
\[ F_y = \frac{dp_y}{dt} = 8 \text{N} \]

A) \( p_x, p_y \) are conserved

B) \( p_x \) conserved, \( p_y \) not conserved

C) \( p_x \) not conserved, \( p_y \) conserved

D) \( p_x, p_y \) not conserved.
• **Example:** one dimensional explosion

A space hauler coupled to a space module moves with velocity $\vec{v}_i = v_i \hat{i}$ relative to the Sun. The total mass of the system is $M$ and the mass of the module is $m < M$.

The module is ejected by a small explosion such that the relative velocity of the hauler with respect to the module is $\vec{v}_{rel} = v_{rel} \hat{i}$.

Find the velocity of the hauler $\vec{v}_{HS}$ relative to the Sun after the explosion.
Isolated closed system: \( \vec{P} \) conserved.

\[
\vec{P}_i = \vec{P}_f \quad P_{ix} = P_{fx}
\]

\[
P_{ix} = Mv_{ix} \quad P_{fx} = (M - m)v_{HS,x} + mv_{MS,x}
\]

\[
\vec{v}_{HS} = \vec{v}_{MS} + \vec{v}_{rel} \quad v_{HS,x} = v_{MS,x} + v_{rel,x}
\]
\[(M - m)v_{HS,x} + mv_{MS,x} = Mv_{ix}\]

\[v_{MS,x} = v_{HS,x} - v_{rel,x}\]

\[\begin{align*}
(M - m)v_{HS,x} + m(v_{HS,x} - v_{rel,x}) &= Mv_{ix} \\
v_{HS} &= v_{ix} + \frac{m}{M}v_{rel,x}
\end{align*}\]
**Example:** two dimensional explosion

A firecracker placed inside a coconut of mass $M$, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor.

Piece $C$, with mass $M_C = 0.3M$, has final speed $v_{fC} = 5.0\text{ m/s}$.

(a) What is the speed of piece $B$, which has mass $M_B = 0.20M$?

(b) What is the speed of $A$?
Isolated closed system:

\[ \vec{P}_i = \vec{P}_f \]

\[ 0 = M_A \vec{v}_{fA} + M_B \vec{v}_{fB} + M_C \vec{v}_{fC} \]

\[ 0 = M_A v_{fA,x} + M_B v_{fB,x} + M_C v_{fC,x} \]

\[ 0 = M_A v_{fA,y} + M_B v_{fB,y} + M_C v_{fC,y} \]

x-axis: \[ M_A v_{fA} = M_B v_{fB} \cos \theta_B + M_C v_{fC} \cos \theta_C \]

y-axis: \[ 0 = M_B v_{fB} \sin \theta_B - M_C v_{fC} \sin \theta_C \]
\[ M_A v_{fA} = M_B v_{fB} \cos \theta_B + M_C v_{fC} \cos \theta_C \]

\[ M_B v_{fB} \sin \theta_B = M_C v_{fC} \sin \theta_C \]

\[ v_{fB} = \frac{M_C v_{fC} \sin \theta_C}{M_B \sin \theta_B} \]

\[ v_{fA} = \frac{M_B}{M_A} v_{fB} \cos \theta_B + \frac{M_C}{M_A} v_{fC} \cos \theta_C \]
A golf ball of mass $m$ moving with speed $v_0$ hits a bowling ball initially at rest. The golf ball bounces back with speed $0.9v_0$. Let $p_B$ denote the magnitude of the momentum of the bowling ball after collision. Which of the following statements is true?

- $p_B = mv_0$
- $p_B = 0.9mv_0$
- $p_B = 0.1mv_0$
- $p_B = 1.9mv_0$
Answer

A golf ball of mass $m$ moving with speed $v_0$ hits a bowling ball initially at rest. The golf ball bounces back with speed $0.9v_0$. Let $p_B$ denote the magnitude of the momentum of the bowling ball after collision. Which of the following statements is true?

$$A) \quad p_B = mv_0 \quad B) \quad p_B = 0.9mv_0$$

$$C) \quad p_B = 0.1mv_0 \quad D) \quad p_B = 1.9mv_0$$
\[ \vec{P}_i = \vec{P}_f \Rightarrow m v_0 = -0.9 m v_0 + p_{B,x} \]

\[ p_{B,x} = 1.9 m v_0 \]
● Momentum and kinetic energy in collisions

Collisions in closed isolated system

● kinetic energy conserved ⇒ elastic collisions

● kinetic energy not conserved, transferred to other forms of energy such as thermal energy ⇒ inelastic collisions
**Completely inelastic collisions in 1D**

**Completely inelastic:** the objects stick together after collision.
Here is the generic setup for an inelastic collision.

\[ \vec{P}_i = \vec{P}_f \]

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \]
In a completely inelastic collision, the bodies stick together.

\[ \vec{P}_i = \vec{P}_f \]

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{V} \]

\[ \vec{V} = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2} \]

\[ V_x = \frac{m_1 v_{1i,x} + m_2 v_{2i,x}}{m_1 + m_2} \]
The com of the two bodies is between them and moves at a constant velocity.

\[ \vec{V} = \vec{v}_{\text{com}} \]

Here is the incoming projectile.

Here is the stationary target.

Collision!

The com moves at the same velocity even after the bodies stick together.
An object at rest explodes into two pieces of unequal mass. One piece flies west at a speed \( v \) and the second flies east at a speed \( 3v \). What is the velocity of the center of mass?

\[ A) \ 0 \quad D) \text{cannot be determined} \]
\[ B) \ 2v \ \text{west} \]
\[ C) \ 2v \ \text{east} \]
i-Clicker

An object at rest explodes into two pieces of unequal mass. One piece flies west at a speed $v$ and the second flies east at a speed $3v$. What is the velocity of the center of mass?

$\vec{P}_i = \vec{P}_f = M\vec{v}_{\text{com}} = 0$

A) 0  
B) $2v$ west  
C) $2v$ east  
D) cannot be determined
• Example: ballistic pendulum

- large block of wood of mass $M = 5.4\,\text{kg}$ suspended from two long cords.
- bullet of mass $m = 9.5\,\text{g}$ fired into the block.
- system block + bullet swings upward a vertical distance $h = 6.3\,\text{cm}$.
- $v_{\text{bullet}}$ ?
**Step 1:**
- completely inelastic collision
- linear momentum conserved
- kinetic energy not conserved

\[ mv = (m + M)V \]
Step 2: • upward swing
  • linear momentum not conserved
  • mechanical energy conserved

\[ E_{\text{mec}} = K + U = \text{constant} \]

\[ \frac{1}{2}(M + m)V^2 = (M + m)gh \]

\[ V = \sqrt{2gh} \]

\[ v = \frac{m + M}{m} \sqrt{2gh} \]
**Example:** generic inelastic collision

A bullet of mass \( m = 10\, \text{g} \) moving directly upward at \( v = 1000\, \text{m/s} \) strikes and passes through the center of mass of a \( M = 5.0\, \text{kg} \) block initially at rest.

The bullet emerges from the block moving directly upward at \( v_1 = 400\, \text{m/s} \).

To what maximum height does the block then rise above its initial position?
Step 1: • inelastic collision  
• linear momentum conserved  
• kinetic energy not conserved

\[ \vec{P}_i = \vec{P}_f \]

\[ mv = mv_1 + Mv_2 \]

\[ v_2 = \frac{m}{M}(v - v_1) \]
Step 2: • upward motion
• linear momentum not conserved
• mechanical energy conserved
• assume the block does not move much during the collision

\[
\frac{1}{2} M v_2^2 = M g h
\]

\[
h = \frac{v_2^2}{2g} = \frac{m^2}{2M^2g} (v - v_1)^2
\]
- **Elastic collisions in 1D: both** linear momentum and kinetic energy are conserved.
• **Generic setup – stationary target**

- The linear momentum of the systems is conserved:
  \[ \vec{P}_i = \vec{P}_f \]

- The total kinetic energy of the system is conserved:
  \[ K_i = K_f \]

**Note:** the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.
Here is the generic setup for an elastic collision with a stationary target.

Before

\[ \vec{v}_{1i} \]
\[ \vec{v}_{2i} = 0 \]

Projectile \hspace{1cm} Target

\[ m_1 \hspace{1cm} m_2 \]

After

\[ \vec{v}_{1f} \]
\[ \vec{v}_{2f} \]

\[ m_1 \hspace{1cm} m_2 \]

\[ v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \]
\[ v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} \]

\[ m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \]

\[ \frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]
1D elastic collision – stationary target

\[ v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} \]

Note:

- \( v_{2f} > 0 \)
- \( v_{1f} > 0 \) if \( m_1 > m_2 \); \( v_{1f} < 0 \) if \( m_1 < m_2 \)
- \( v_{1f} = 0, \; v_{2f} = v_{1i} \) if \( m_1 = m_2 \) (identical particles)
• Generic setup – moving target

Here is the generic setup for an elastic collision with a moving target.

\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]

\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]
\begin{align*}
m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\
\Rightarrow m_1 (v_{1i} - v_{1f}) &= m_2 (v_{2f} - v_{2i}) \\
\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 &= \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \\
\Rightarrow m_1 (v_{1i} - v_{1f}) (v_{1i} + v_{1f}) &= m_2 (v_{2f} - v_{2i}) (v_{2f} + v_{2i}) \\
\Rightarrow v_{1i} + v_{1f} &= v_{2f} + v_{2i}
\end{align*}
1D elastic collision – moving target

\[ v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \]

\[ v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \]
• **Example:** two pendulums

- How high will the 1st ball recoil after collision?
- Which way will it swing?
- How high will the 2nd ball swing after collision?
- **Step 1:**

  \[mg h_1 = \frac{1}{2} m v_{1i}^2\]
  \[v_{1i} = \sqrt{2g h_1}\]

- **Step 2:** collision

  \[v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}\]
  \[v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}\]

- **Step 3:**

  \[m_1 g h_{1f} = \frac{1}{2} m_1 v_{1f}^2\]
  \[m_2 g h_{2f} = \frac{1}{2} m_2 v_{2f}^2\]
• **Collisions in 2D**

A glancing collision that conserves both momentum and kinetic energy.

![Diagram of 2D collision](image)

- **Linear momentum conserved:**
  \[ \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \]

- **Stationary target:**
  \[
  m_1 v_{1i} = m_1 v_{1f} \cos \theta_1 \\
  + m_2 v_{2f} \cos \theta_2 \\
  m_1 v_{1f} \sin \theta_1 = m_2 v_{2f} \sin \theta_2
  \]

- **If elastic**, kinetic energy also conserved:
  \[
  \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2
  \]
• Systems with varying mass

(a) accelerating rocket at time $t$ in inertial frame
(b) accelerating rocket at time $t + dt$ in the same frame

$v \rightarrow v + dv, \quad dv > 0 \quad M \rightarrow M + dM, \quad dM < 0
• Suppose the relative speed $v_{\text{rel}}$ between the rocket and exhaust products is known.

• How do we find the acceleration?
Rocket + exhaust products = isolated closed system

\[ \vec{P}_a = \vec{P}_b \quad P_{a,x} = P_{b,x} \]
\[ P_{a,x} = Mv \quad P_{b,x} = (M + dM)(v_x + dv_x) + (-dM)u_x \]

**Note:** \( u_x \) the \( x \)-component of the velocity of the exhaust products relative to the inertial frame

\[ v_x + dv_x = u_x + v_{rel} \]
The ejection of mass from the rocket's rear increases the rocket's speed.

\[ Mv = (M + dM)(v_x + dv_x) - (v_x + dv_x - v_{rel})dM \]

\[ Mdv_x + v_{rel}dM = 0 \quad \Rightarrow \quad M\frac{dv_x}{dt} = -v_{rel}\frac{dM}{dt} \]
The ejection of mass from the rocket's rear increases the rocket's speed.

(a) \[ M \rightarrow v \]

(b) \[ -dM \rightarrow u \]

\[ M a_x = -v_{\text{rel}} \frac{dM}{dt} = R v_{\text{rel}} \]

**The 1st rocket equation**
The ejection of mass from the rocket's rear increases the rocket's speed.

\[ v_{f,x} - v_{i,x} = v_{rel} \ln \frac{M_i}{M_f} \]

The 2nd rocket equation
Rain falls vertically into an open cart rolling horizontally. What happens to the momentum, speed and kinetic energy?

<table>
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<th></th>
<th>$p$</th>
<th>$v$</th>
<th>$K$</th>
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<td>same</td>
<td>same</td>
</tr>
<tr>
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<td>same</td>
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<tr>
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</tr>
<tr>
<td>$E$</td>
<td>same</td>
<td>decreases</td>
<td>decreases</td>
</tr>
</tbody>
</table>
Rain falls vertically into an open cart rolling horizontally. What happens to the momentum, speed and kinetic energy?

\[ \begin{array}{ccc} 
A) & \text{same} & \text{same} & \text{same} \\
B) & \text{increases} & \text{same} & \text{increases} \\
C) & \text{increases} & \text{increases} & \text{increases} \\
D) & \text{same} & \text{decreases} & \text{same} \\
E) & \text{same} & \text{decreases} & \text{decreases} \\
\end{array} \]