Honors Analytical Physics I

Lecture 11
• **Thanksgiving week, Nov 20-24**: No homework due, no recitations, no class.

• **Final Exam**: Tuesday, December 19th, 4-7pm (PHL?)
Law of Conservation of Energy

\[ W_{\text{ext}} = \Delta E = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}} \]

- \( \Delta E \) is the change in the **total** energy of the system.
- \( W_{\text{ext}} \) net work done by **external** forces
- \( \Delta E_{\text{mec}} \) is any change in the mechanical energy of the system,
- \( \Delta E_{\text{th}} \) is any change in the thermal energy of the system,
- \( \Delta E_{\text{int}} \) is any change in any other type of internal energy of the system.
• **Conservative and dissipative forces inside the system:**

\[ W_{\text{ext}} = \Delta U + \Delta K + |W_{\text{dissipative}}| \]

\[ W_{\text{ext}} = \Delta E_{\text{mec}} + |W_{\text{dissipative}}| \]

**Note:** work of friction or drag forces

\[ \Downarrow \]

heat

\[ \Downarrow \]

Increase in **thermal** energy

\[ |W_{\text{dissipative}}| = \Delta E_{\text{th}} \]
• **Isolated** system: \( \vec{F}_{\text{ext}} = 0, \ W_{\text{ext}} = 0 \).

The total energy \( E \) of an isolated system cannot change.

\[
\Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}} = 0 \quad \text{(isolated system)}
\]
**Example**: an object slides along a frictionless floor with speed $v_1$. It hits a relaxed spring placed on rough surface such that the kinetic friction is $f_k$. What is the distance $d$ when the object stops?
Isolated system:

\[ \Delta E_{\text{mec}} + \Delta E_{\text{th}} = 0 \]

\[ (E_{\text{mec}})_i = \frac{mv_1^2}{2} \quad (E_{\text{mec}})_f = \frac{kd^2}{2} \]

\[ \Delta E_{\text{mec}} = \frac{kd^2}{2} - \frac{mv_1^2}{2} \]

\[ \Delta E_{\text{th}} = f_k d \]

\[ \frac{kd^2}{2} + f_k d - \frac{mv_1^2}{2} = 0 \]
• A block slides along a track from one level to a higher level after passing through an intermediate valley.

• The track is frictionless until the block reaches the higher level. There a frictional force stops the block in a distance $d$.

• Given $v_0$ and $h$, find $d$. 
• Block + track + Earth isolated system.

\[ \Delta E = 0, \quad E = K + U_g + E_{th} \]

\[ \Delta K = -\frac{mv_0^2}{2}, \quad \Delta U_g = +mgh \]

\[ \Delta E_{th} = +\mu_k m gd \]
\[- \frac{mv_0^2}{2} + mgh + \mu_k mgd = 0\]

\[d = \frac{v_0^2 - 2gh}{2\mu_k g}\]

- \(v_0^2 < 2gh\) \(\Rightarrow\) the block does not reach the higher level.
• A block slides along a frictionless path until it reaches the section of length $L = 0.75 \text{ m}$, which begins at height $h = 2.0 \text{ m}$ on a ramp of angle $\theta = 30^\circ$.
  • $v_A = 8.0 \text{ m/s}$.
  • If the block can reach point $B$, what is its speed there?
  • If not what is its greatest height above $A$?
• **Isolated** system: block + track + Earth

\[ \Delta E_{AB} = \Delta K_{AB} + \Delta U_{AB} + \Delta E_{th} = 0 \]

\[ \Delta K_{AB} = \frac{mv_B^2}{2} - \frac{mv_A^2}{2} \quad \Delta U_{AB} = mgh + mgL\sin(\theta) \]

\[ \Delta E_{th} = \mu_k mgL\cos(\theta) \]
\[
\frac{mv_B^2}{2} - \frac{mv_A^2}{2} + mgh + mgL\sin(\theta) + \mu_k mgL\cos(\theta) = 0
\]

\[
v_B^2 = v_A^2 - 2gh - 2gL\sin(\theta) - 2\mu_k gL\cos(\theta)
\]

The block reaches \( B \) with speed:

\[
v_B^2 = 14.90388531 \ldots > 0
\]
Suppose \( v_A = 7.0 \text{ m/s} \).

\[
v_A^2 - 2gh - 2gL\sin(\theta) - 2\mu_kgL\cos(\theta) = -0.096114688\ldots < 0
\]

The block does not reach \( B \). Stops at some point \( C \) between \( A \) and \( B \).
Suppose \( v_A = 7.0 \) m/s.

\[
v_A^2 - 2gh - 2gL\sin(\theta) - 2\mu_kgL\cos(\theta) = -0.096114688 \ldots < 0
\]

\[
v_A^2 - 2gh = 9.8 > 0
\]

The point \( C \) is within the rough section of the road. Where?
\[ x = \text{distance between } C \text{ and the lowest point of the rough section along the ramp} \]

\[ v_A^2 - 2gh - 2gx\sin(\theta) - 2\mu_kgx\cos(\theta) = 0 \]

\[ x = \frac{v_A^2 - 2gh}{2g\sin(\theta) + 2\mu_kg\cos(\theta)} = 0.7427157257\ldots \]
9. Center of Mass. Linear Momentum I

The center of mass of a system of particles is the point that moves as though:

(1) all mass of the system were concentrated there and

(2) all external forces applied there.

The center of mass of the baseball bat follows a parabolic path, but all other points follow more complicated paths.
• The center of mass: a system of particles

Discrete linear distribution of particles

Note: in general, the center of mass does not have to be one of the particles in the system.

\[
x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M} \sum_{i=1}^{N} m_i x_i
\]
- **Three dimensions**: discrete distribution of particles

\[
\begin{align*}
\vec{r}_{\text{CM}} &= \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i \\
\vec{x}_{\text{com}} &= \frac{1}{M} \sum_{i=1}^{N} m_i x_i \\
\vec{y}_{\text{com}} &= \frac{1}{M} \sum_{i=1}^{N} m_i y_i \\
\vec{z}_{\text{com}} &= \frac{1}{M} \sum_{i=1}^{N} m_i z_i \\
\vec{r}_{\text{com}} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_N \vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i
\end{align*}
\]
**Solid body**: continuous mass distribution

\[ dm = \rho \, dv \]

\[ \vec{r}_{\text{com}} = x_{\text{com}} \hat{i} + y_{\text{com}} \hat{j} + z_{\text{com}} \hat{k} \]

\[ x_{\text{com}} = \frac{1}{M} \int x \, dm \quad y_{\text{com}} = \frac{1}{M} \int y \, dm \quad z_{\text{com}} = \frac{1}{M} \int z \, dm \]

\[ M = \int dm \quad \text{total mass of the object} \]
Mass density:

\[ \rho = \frac{dm}{dv} \quad dv = dx
dy
dz \quad \text{volume element} \]

**Uniform** mass distribution:

\[ \rho = \text{constant} \quad \text{as function of } \vec{r} \]

\[ \downarrow \]

\[ x_{\text{com}} = \frac{1}{V} \int x dv \quad y_{\text{com}} = \frac{1}{V} \int y dv \quad z_{\text{com}} = \frac{1}{V} \int z dv \]

\[ V = \int dv = \int dx
dy
dz \quad \text{total volume} \]
• **Symmetry:** if the object has a symmetry axis and $\rho = \text{constant}$ then its com must lie on that axis
Consider an L shape made up of 4 blocks each a square with sides of length $d$.

- Assume all blocks are uniform and have the same mass $M$.
- Treat this as a 2-dimensional problem – ignore width in the $z$-direction.

$$x_{\text{com}} = \frac{3M(d/2) + M(3d/2)}{4M} = \frac{3}{4}d$$
Consider an L shape made up of 4 blocks each a square with sides of length d.

- Assume all blocks are uniform and have the same mass $M$
- Treat this as a 2-dimensional problem - ignore width in the $z$-direction.

What is $y_{com}$?

A) $d$  B) $7d/8$  C) $5d/4$  D) $3d/4$  E) $d/2$
Answer

Consider an L shape made up of 4 blocks each a square with sides of length $d$.

- Assume all blocks have the same mass $M$
- Treat this as a 2-dimensional problem ignore width in the $z$-direction.

What is $y_{\text{com}}$?

A) $d$  B) $7d/8$  C) $5d/4$  D) $3d/4$  E) $d/2$
\[ y_{\text{com}} = \frac{2M(d/2) + M(3d/2) + M(5d/2)}{4M} = \frac{5}{4}d \]
Example:

Consider the two uniform squares shown each with sides of length \( d \).

- Assume \( m_{\text{blue}} = 2 \text{ kg} \), \( m_{\text{orange}} = 1 \text{ kg} \)
- Treat this as a 2-dimensional problem.

\[
x_{\text{com}} = \frac{m_{\text{blue}}x_{\text{blue}} + m_{\text{orange}}x_{\text{orange}}}{m_{\text{blue}} + m_{\text{orange}}} = -\frac{d}{6}
\]
Consider the two squares shown each with sides of length d.

- Assume \( m_{\text{blue}} = 2 \text{ kg} \), \( m_{\text{orange}} = 1 \text{ kg} \)
- Treat this as a 2-dimensional problem. What is \( y_{\text{com}} \)?

\[
\begin{align*}
A) & \ d & B) & \ d/4 & C) & \ 0 & D) & 3d/2 & E) & d/2 \\
\end{align*}
\]
Answer

Consider the two squares shown each with sides of length $d$.

- Assume $m_{\text{blue}} = 2 \text{ kg}$, $m_{\text{orange}} = 1 \text{ kg}$
- Treat this as a 2-dimensional problem.

What is $y_{\text{com}}$?

$A) \ d \quad B) \ d/4 \quad C) \ 0 \quad D) \ 3d/2 \quad E) \ d/2$
\[ y_{\text{com}} = \frac{m_{\text{blue}}y_{\text{blue}} + m_{\text{orange}}y_{\text{orange}}}{m_{\text{blue}} + m_{\text{orange}}} = \frac{d}{2} \]
Subtraction example

A metal plate $P$ is has the shape of a disk of radius $R$ with a hole of radius $R$. Assuming uniform density, $\rho = \text{constant}$, find $x_{\text{com}}$.

Treat the problem as two-dimensional ignoring thickness in the $z$-direction.
- $P$: holed disk
- $D$: whole big disk
- $H$: the small disk removed from $D$

\[ P = D - H \]

\[ x^P_{\text{com}} = \frac{\int_P x \, dx \, dy}{\int_P dx \, dy} = \frac{X_P}{A_P} \]

\[ X_P = \int_D x \, dx \, dy - \int_H x \, dx \, dy = X_D - X_H \]

\[ A_P = \int_D dx \, dy - \int_H dx \, dy = A_D - A_H \]
\[ x^{D}_{\text{com}} = 0 \quad x^{H}_{\text{com}} = -R \]

\[ x^{D}_{\text{com}} = \frac{X_D}{A_D} \quad x^{H}_{\text{com}} = \frac{X_H}{A_H} \]

\[ X_D = 0, \quad X_H = -RA_H \]

\[ x^{P}_{\text{com}} = \frac{X_D - X_H}{A_D - A_H} = R \frac{A_H}{A_D - A_H} = \frac{R}{3} \]
• Newton's 2nd law for a system of particles

\[ \mathbf{F}_{\text{net}} = M \mathbf{a}_{\text{com}} \]

- \( \mathbf{F}_{\text{net}} \) is the net \textit{external} force acting on the system. \textit{Internal} forces are not included.
- \( M \) is the total mass of the system. If \( M \) is constant, the system is said to be \textit{closed}.
- \( \mathbf{a}_{\text{com}} \) is the acceleration of the center of mass of the system.

\[ F_{\text{net},x} = M a_{\text{com},x} \]
\[ F_{\text{net},y} = M a_{\text{com},y} \]
\[ F_{\text{net},z} = M a_{\text{com},z} \]
The internal forces of the explosion cannot change the path of the com.

Fig. 9-5 A fireworks rocket explodes in flight. In the absence of air drag, the center of mass of the fragments would continue to follow the original parabolic path, until fragments began to hit the ground.
Derivation

\[ M \vec{r}_{\text{com}} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_N \vec{r}_N \]
\[ \downarrow \frac{d}{dt} \]

\[ M \vec{v}_{\text{com}} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots + m_N \vec{v}_N \]
\[ \downarrow \frac{d}{dt} \]

\[ M \vec{a}_{\text{com}} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + \cdots + m_N \vec{a}_N \]

\[ M \vec{a}_{\text{com}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N \]
\[ M \vec{a}_{\text{com}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N \]

\( \vec{F}_i \) is the sum of all forces acting on particle \( i \), including internal ones.

\[ \vec{F}_i = \vec{F}_{i,\text{ext}} + \vec{F}_{i,\text{int}} \]

**Newton’s 3rd law:**

\[
\sum_{i=1}^{N} \vec{F}_{i,\text{int}} = 0
\]

\[ M \vec{a}_{\text{com}} = \vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}} + \cdots + \vec{F}_{N,\text{ext}} \]
The com of the system will move as if all the mass were there and the net force acted there.

\[ M \mathbf{\dot{a}}_{\text{com}} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \]

\[ M = m_1 + m_2 + m_3 \]
Problem 12. Ch. 9. Two skaters, one with mass 65 kg and the other with mass 40 kg, stand on an ice rink holding a pole of length 10 m and negligible mass. Starting from the ends of the pole, the skaters pull themselves along the pole until they meet. How far does the 40 kg skater move? Neglect friction.

\[ \vec{F}_{\text{ext}} = 0 \Rightarrow \vec{a}_{\text{com}} = 0 \text{ and } \vec{v}_{\text{com}} = 0. \] Hence the two skaters meet at the COM, which does not move.

Initially: \[ x_1 = 0 \quad x_2 = d \]

\[ x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_2 d}{m_1 + m_2} = \Delta x_1 \]
Suppose the kinetic friction coefficient between the skates and the ice is $\mu_k$. What is the acceleration of the center of mass?

**External** force:

$$\vec{F}_{\text{ext}} = \vec{f}_{k1} + \vec{f}_{k2}$$

$$= \mu_k (m_2 - m_1) g \hat{i}$$

$$a_{\text{com},x} = \frac{m_2 - m_1}{m_1 + m_2} \mu_k g$$
Linear momentum

How can we predict the outcome of a collision?

Assuming energy is conserved

\[ K_1 + K_2 = K_1' + K_2' \]

Not sufficient!

Energy is a scalar quantity.

No direction!
Linear momentum of a particle

\[ \vec{p} = m\vec{v} \]

Momentum \( \vec{p} \) is a vector quantity; a particle’s momentum has the same direction as its velocity \( \vec{v} \).
The rate of change of the momentum is equal to the net force acting on the particle:

\[ \frac{d\vec{p}}{dt} = \vec{F}_{\text{net}} \]

\[ \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}_{\text{net}} \]
• **Linear momentum of a system of particles**

The linear momentum of a system of particles is equal to the product of the total mass $M$ of the system and the velocity of the center of mass.

\[
\vec{P} = M \vec{v}_{\text{com}}
\]

\[
\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i = M \vec{v}_{\text{com}}
\]
The rate of change of the momentum is equal to the net external force acting on the system:

\[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \]
Collision and Impulse

The figure depicts the collision at one instant. The ball experiences a force $F(t)$ that varies during the collision and changes the linear momentum of the ball.
The change in linear momentum is related to the force by Newton’s second law:

\[ \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \Delta \vec{p} = \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F} \, dt \]

**Impulse:**

\[ \vec{J} = \int_{t_i}^{t_f} \vec{F} \, dt \]
• The magnitude of $\vec{J}$ equals the area under the curve $F(t)$.

• Average force

$$\vec{F}_{\text{average}} = \frac{\vec{J}}{\Delta t}$$

• Newton’s 3rd law:

$$\vec{F}_{\text{ball}}(t) + \vec{F}_{\text{bat}}(t) = 0$$

at all times. Hence:

$$\vec{J}_{\text{bat}} = -\vec{J}_{\text{ball}}$$

$$|\vec{J}_{\text{bat}}| = |\vec{J}_{\text{ball}}|$$
Consider two ways to slow a toy car from 15km/h to a complete stop.

(i) The car slams into a wall and stops.

(ii) The car gradually plows into a tube of gelatine and comes to a gradual halt.

In which case is the magnitude of the impulse bigger?

A) Case (i)          B) Case (ii)
C) They are equal.    D) Cannot be decided.
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In which case is the magnitude of the impulse bigger?

A) Case (i)  
B) Case (ii)  
C) They are equal.  
D) Cannot be decided.

\[ \vec{J} = \Delta \vec{p}, \]  
the same in both cases.