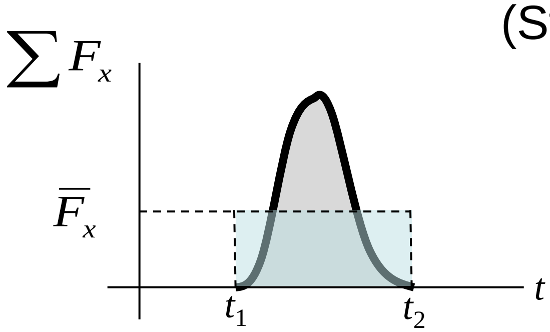
CENTER OF MASS MOTION & ROTATIONAL KINEMATICS

REVIEW

- LINEAR MOMENTUM: $\vec{p} = m\vec{v}$ $\quad \Sigma\vec{F} = m\vec{a} = \dfrac{d\vec{p}}{dt}$

- IMPULSE OF A FORCE: $\vec{J} = \overline{\Sigma\vec{F}}(t_2 - t_1)$

(Strong force acting for short time)



$$J_x = \int_{t_1}^{t_2} F_x dt = \bar{F}_x (t_2 - t_1)$$

- Impulse-Momentum Theorem

$$\vec{J} = \vec{p}_2 - \vec{p}_1$$

- CONSERVATION OF LINEAR MOMENTUM:

$$\text{If} \quad \Sigma F_{ext} = 0 \quad \text{then} \quad \vec{P}_F = \vec{P}_I \Rightarrow [\vec{p}_{A_I} + \vec{p}_{B_I}] = [\vec{p}_{A_F} + \vec{p}_{B_F}]$$

- TYPES OF COLLISIONS:

  Elastic: $\vec{P}$ & KE conserved

  Inelastic: $\vec{P}$ only is conserved

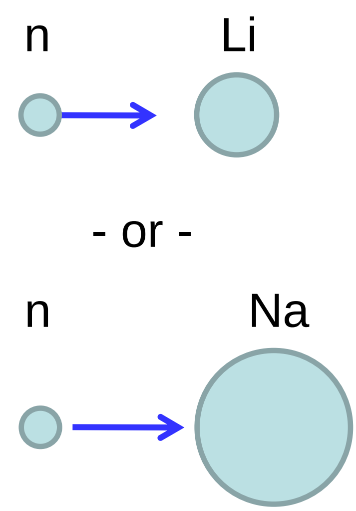
  Completely Inelastic: $\vec{P}$ only conserved & objects stick

- LINEAR MOMENTUM CONSERVED IN COLLISIONS

(When collision force >> external force)

i-Clicker

Many nuclear reactors use Li ($M = 7m_o$) to moderate (reduce the KE) of neutrons ($M_n = m_o$) produced in nuclear reactions by elastic collisions with Li nuclei.. An alternative is Na ($M_{Na} = 23\ m_o$).



Li is a better choice because:

A. P_{tot} is lower after $n \rightarrow$ Li collision

B. n has less KE after $n \rightarrow$ Li collision compared to $n \rightarrow$ Na collision

C. P_{tot} is higher after $n \rightarrow$ Na collision

D. KE of n does not change after $n \rightarrow$ Na collision

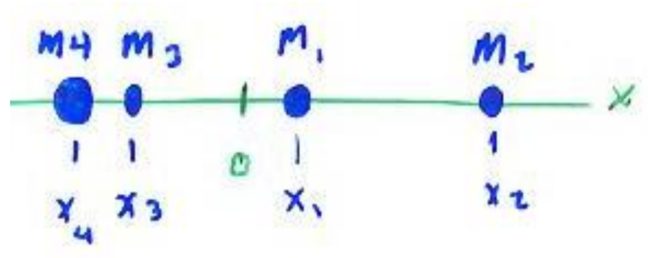
E. Na is better choice (trick question)

$$P_{final} = P_{initial}$$

CENTER OF MASS

Point in an extended mechanical system that moves as though all the mass were concentrated at that point

Consider a collection of (different) masses distributed on x-axis.



Define center of mass:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ \dots}{m_1 + m_2 + m_3 + \ \dots}$$

i.e.:

$$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M}$$

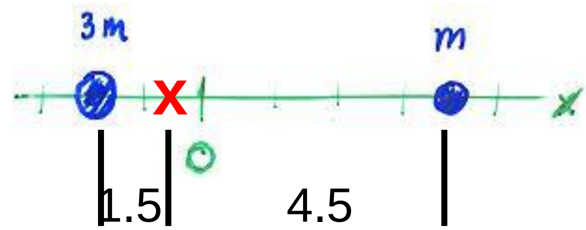
Similar expression for y_{cm} and z_{cm}

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{M}$$

If mass is a continuous distribution:

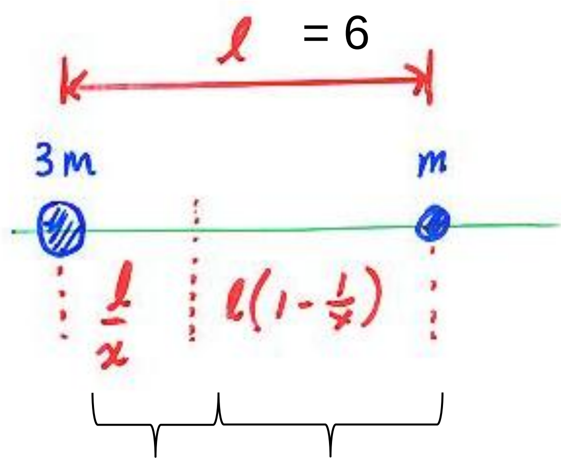
$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r}\, dm$$

EXAMPLE:



$$x_{cm} = \frac{\cancel{m}(4) + 3\cancel{m}(-2)}{\cancel{m} + 3\cancel{m}}$$

$$= \frac{(4+(-6))}{4} = \frac{-2}{4} = -\frac{1}{2}$$



$$3\cancel{m}\left(\frac{\cancel{l}}{x}\right) = \cancel{m}\cancel{l}\left(1 - \frac{1}{x}\right)$$

$$\frac{l}{4} \qquad \frac{3l}{4}$$

$$\frac{3}{x} = \left(1 - \frac{1}{x}\right)$$

$$\frac{6}{4} \qquad \frac{18}{4}$$

$$\frac{4}{x} = 1 \Rightarrow x = 4$$

$$1\frac{1}{2} \qquad 4\frac{1}{2}$$

x_{cm} is mass-weighted average position of the particles

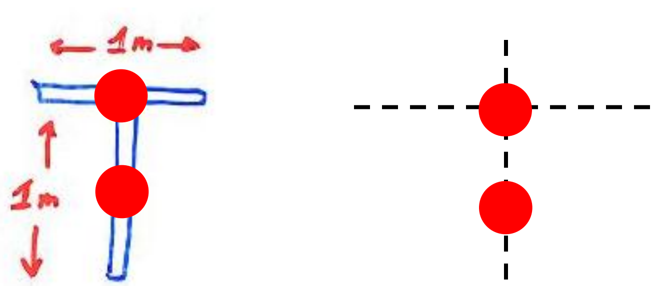
LOCATING CM

• If body is homogeneous and has a geometric center (e.g., uniform sphere or cube) then CM is geometric center

• CM of symmetric body is along axis of symmetry (cylinder, wheel, dumbbell)

• For collection of extended bodies, find CM of each body, then CM of collection is CM of those point masses

EXAMPLE: Two meter sticks, each with a mass of 0.25 kg, form a “T”. Where is the CM?



$$x_{\text{cm}} = 0$$

$$y_{\text{cm}} = \frac{(0.25\,\text{kg})0 + (0.25\,\text{kg})(-0.5\,\text{m})}{(0.5\,\text{kg})} = -0.25\,\text{m}$$

MOTION OF CM

$$\vec{r}_{cm} = \frac{\Sigma_i m_i \vec{r}_i}{M} \quad \Rightarrow M\vec{r}_{cm} = \sum_i m_i \vec{r}_i$$

$$\left(\frac{d}{dt}\right) \Rightarrow M\vec{v}_{cm} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{P}$$

$$\text{again} \left(\frac{d}{dt}\right) \Rightarrow M\vec{a}_{cm} = \frac{d\vec{P}}{dt} = \sum \vec{F}_{ext}$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} = \frac{d\vec{P}}{dt} \quad \cancel{\left(+\sum \vec{F}_{\text{int}}\right)}$$

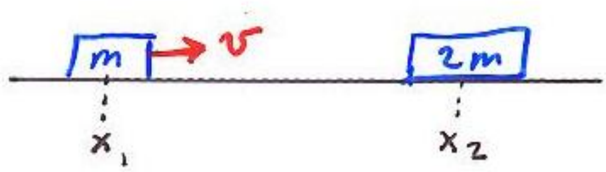
- Under influence of external force CM moves like an imaginary particle of mass M.

- Total momentum $\vec{P}$ of collection of masses can only be changed by external force.

EXAMPLE: Mass m moves along x-axis with speed v towards mass 3m which is at rest.

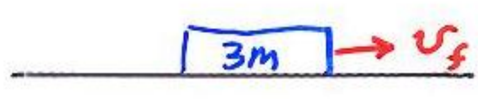
They collide and stick.

What is the motion of the CM before and after the collision?



Before

After



No External Forces: $P_F = P_I$

$$3mv_f = mv_i \Rightarrow v_f = \frac{v_i}{3}$$

Clearly, after collision CM at center of composite mass, $v_{cm} = \frac{v}{3}$

Before collision: $x_{cm} = \dfrac{mx_1 + 2mx_2}{3m} = \dfrac{x_1 + 2x_2}{3}$

$$v_{\text{cm}} = \frac{dx_{\text{cm}}}{dt} = \frac{\frac{dx_1}{dt} + 2\frac{dx_2}{dt}}{3} = \frac{v_i + 2(0)}{3} = \frac{v_i}{3}$$

i-Clicker

A rocket ship moves in a gravity-free region of space with constant velocity. It fires a short burst of gas from the rear engine. Afterwards, the CM of the rocket and gas system has:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$

A) Sped up

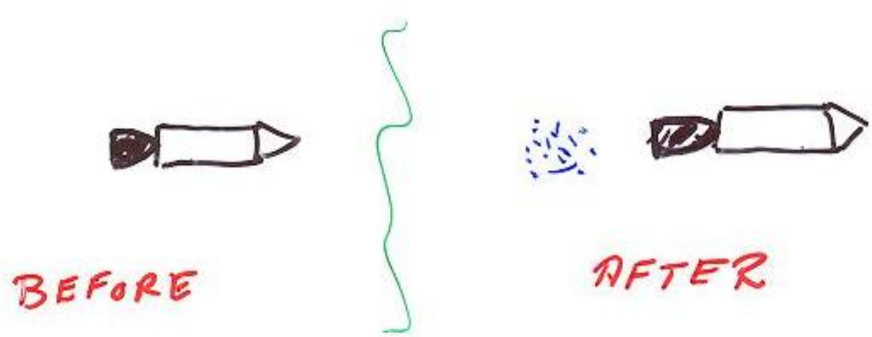
$$\sum \vec{F}_{ext} = 0 \Rightarrow M\vec{a}_{cm} = 0$$

B) Slowed down

C) Has same constant velocity

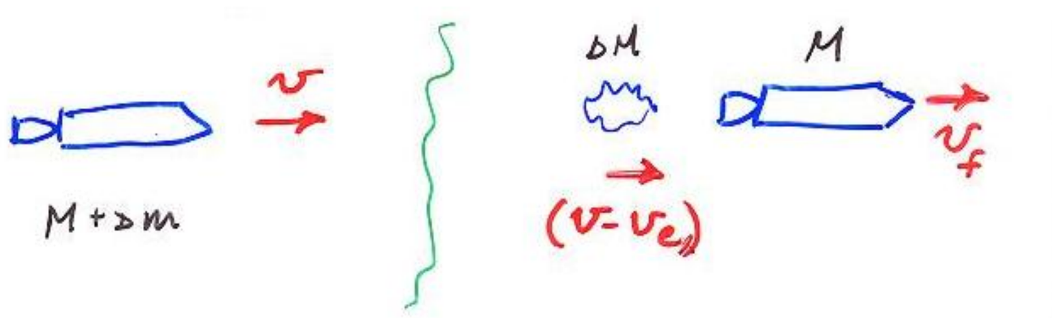
D) Changed but can't tell how

E) Insufficient info to tell anything



ROCKET PROPULSION

Conservation of momentum of rocket and fuel !!



No external forces $\Rightarrow P_F = P_I$

$$P_I = (M + \Delta M)v = P_F = Mv_F + \Delta M(v - v_{ex})$$

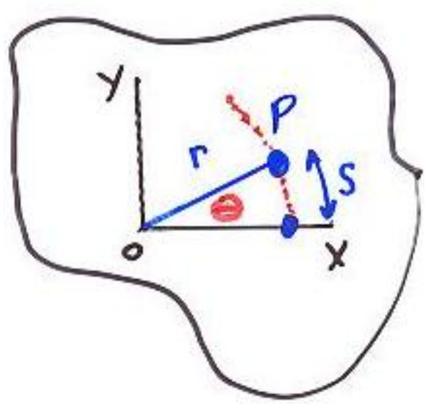
$$Mv = Mv_F - \Delta M v_{ex}$$

$$M(v_F - v) = \Delta M v_{ex}$$

Speed of rocket increases by recoil !

Not from "Pushing on the Air."

RIGID BODY ROTATION



- All points rotate about same axis (through 0,$\perp$ to page)
- If we follow motion of point P as body rotates, only θ changes

→ Use polar coordinates

Measure θ in radians (fractions of 2π)

arc length: $s = r\theta \Rightarrow \theta = \dfrac{s}{r}$ (dimensionless)

$$360^\circ = \frac{2\pi r}{r} = 2\pi \text{ rad} \quad \Rightarrow 1\,\text{rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

$$\text{or :} \quad \theta\ [\text{rad}] = \left(\frac{\pi}{180^\circ}\right)\theta\,[\text{deg}]$$

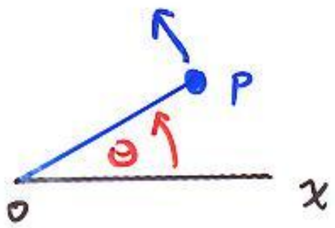
ANGULAR VELOCITY (ω_z)

Rate of change in angle (in xy plane)

Average Angular Velocity: $\omega_{z_{av}} = \left(\dfrac{\theta_2 - \theta_1}{t_2 - t_1}\right) = \dfrac{\Delta\theta}{\Delta t}$

Instantaneous: $\omega_z = \lim_{\Delta t \to 0} \dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt}$

about z - axis



$$\omega_z \quad \begin{matrix} (+) \Rightarrow \text{ccw} \\ (-) \Rightarrow \text{cw} \end{matrix}$$

ANGULAR ACCELERATION (α_z)

- Average: $$\alpha_{z_{av}} = \frac{\omega_{z_2} - \omega_{z_1}}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

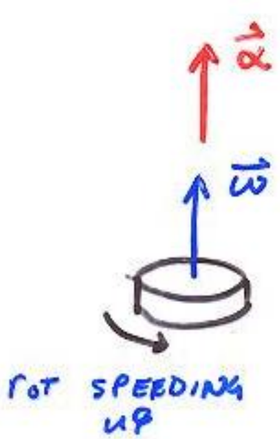
- Instantaneous: $$\alpha_z = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

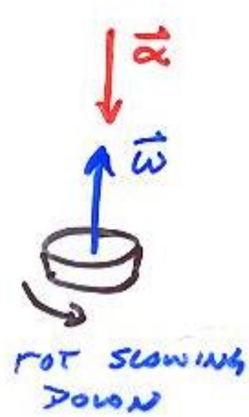
Rigid Body Rotation:

Every particle has the same ω_z and α_z

Define $\vec{\omega}$ and $\vec{\alpha}$ vectors:

- Point along axis of rotation
- Have magnitude of ω and α
- Orientation define by Right Hand Rule





Sign of ω : (+) → ccw
(-) → cw

Sign of α : from $\frac{d\omega}{dt}$

Units of ω: $1\frac{\text{rev}}{\text{s}} = 2\pi\frac{\text{rad}}{\text{s}}$

RIGID BODY ROTATION WITH CONSTANT ANGULAR ACCELERATION

Form of equations same as 1-D motion:

$$x(t) \longleftrightarrow \theta(t)$$

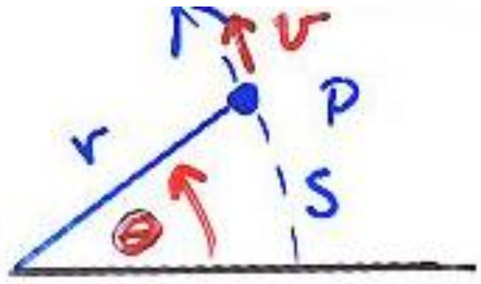
$$v_x = \frac{dx}{dt} \longleftrightarrow \omega_z = \frac{d\theta}{dt}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \longleftrightarrow \alpha_z = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2}$$

Leads to analogous Equations of Motion:

1D:	Fixed Axis Rotation
$a_x = \text{const}$	$\alpha_z = \text{const}$
$v_x = v_{o_x} + a_x t$	$\omega_z = \omega_{oz} + \alpha_z t$
$x = x_o + v_{o_x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_o + \omega_{oz} t + \frac{1}{2} \alpha_z t^2$
$v_x^{\ 2} = v_{o_x}^{\ 2} + 2a_x (x - x_o)$	$\omega_z^{\ 2} = \omega_{oz}^{\ 2} + 2\alpha_z (\theta - \theta_o)$
$x - x_o = \frac{1}{2}(v_x + v_{ox})t$	$\theta - \theta_o = \frac{1}{2}(\omega_z + \omega_{oz})t$

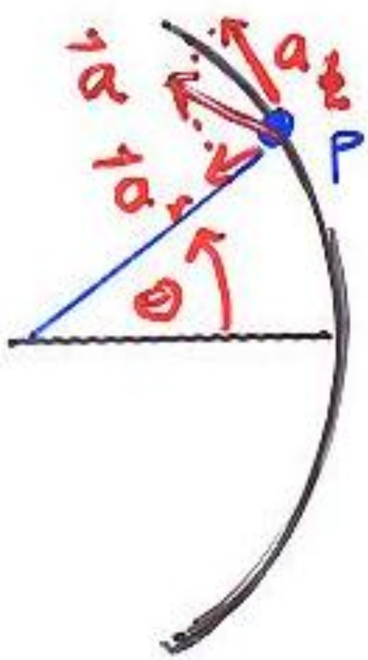
RELATING LINEAR AND ANGULAR MOTION



Each point, P, moves in circle

$$\Rightarrow v = \left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

$$\Rightarrow v = r\omega$$



Magnitude of $\vec{a}_t$

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

Magnitude of centripetal acceleration $\vec{a}_r$

$$a_r = \frac{v^2}{r} = r\omega^2$$

Total acceleration $\vec{a} = \vec{a}_t + \vec{a}_r$

EXAMPLE: A bike tire is rotated by 120º to get the stem vertical. By how many radians was the tire rotated?

$$\Theta_{\mathrm{rad}} = \left(\frac{\pi}{180^\circ}\right)(120^\circ) = \frac{2\pi}{3}$$

EXAMPLE: Your bike tires have a diameter of 0.75 m. You ride from CAC → BC at a speed of 18 km/hr.

What is the angular velocity of your tires?

$$\omega = \left(\frac{v}{r}\right) \qquad v = \left(\frac{18\times10^3\ \mathrm{m/hr}}{3.6\times10^3\ \mathrm{s/hr}}\right) = 5\ \mathrm{m/s}$$

$$\omega = \left(\frac{5\ \mathrm{m/s}}{0.325\ \mathrm{m}}\right) = 13.3\ \mathrm{rad/s}$$

- in rev/s? $\left(\frac{13.3\ \mathrm{rad/s}}{2\pi\ \mathrm{rad/rev}}\right) = 2.12\ \mathrm{rev/s}$

- in RPM? $(2.12\ \mathrm{rev/s})(60\ \mathrm{s/min}) = 127\ \mathrm{rpm}$

EXAMPLE (Cont.): Suppose it took you 4 sec to reach the speed of 18 km/hr. What was the (constant) angular acceleration of your tires during this time?

$$\omega = \not{\omega}_o + \alpha t$$

$$(13.3\,\text{rad/s}) = 0 + \alpha(4\,\text{sec})$$

$$\alpha = 3.33\,\text{rad/s}^2$$

• How many REVs did the tires make while accelerating?

$$\theta = \not{\theta}_o + \not{\omega}_o t + \tfrac{1}{2}\alpha t^2$$

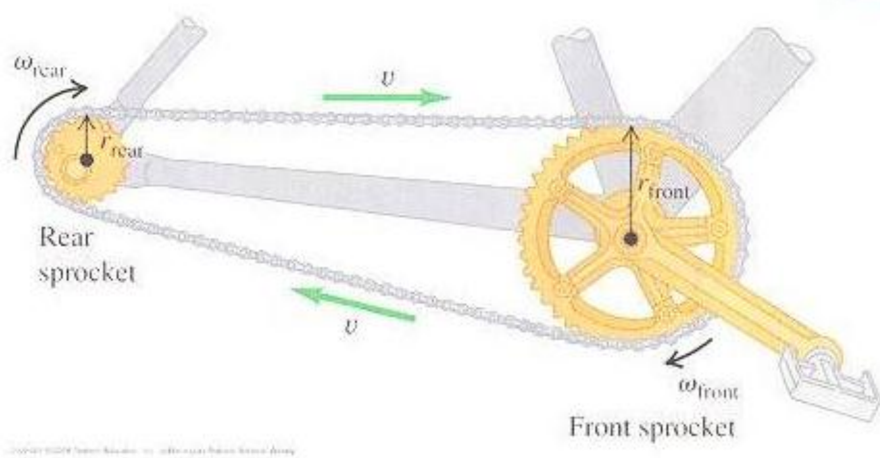
$$\theta = \tfrac{1}{2}(3.33\,\text{rad/s}^2)(4\,\text{s})^2$$

$$\theta = 26.6\,\text{rad}$$

$$\theta = \frac{26.6\,\text{rad}}{2\pi\,\text{rad/rev}} = 4.24\,\text{rev}$$

i-Clicker

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



$$v = r\omega$$

A. a faster linear speed and a faster angular speed.

B. the same linear speed and a faster angular speed.

C. a slower linear speed and the same angular speed.

D. the same linear speed and a slower angular speed.

E. none of the above