P123 Lecture 3 17 Sept 2010

Motion in 2- and 3-Dimensions

REVIEW OF 1-D MOTION

$$\Delta x = x_2 - x_1 \; ; \quad v_{av_x} = \frac{\Delta x}{\Delta t} \; ; \quad v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$a_{x_{av}} = \frac{\Delta v_x}{\Delta t} \qquad a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

- For CONSTANT acceleration in 1-D

$$x = x_o + v_{o_x} t + \tfrac{1}{2} a t^2$$

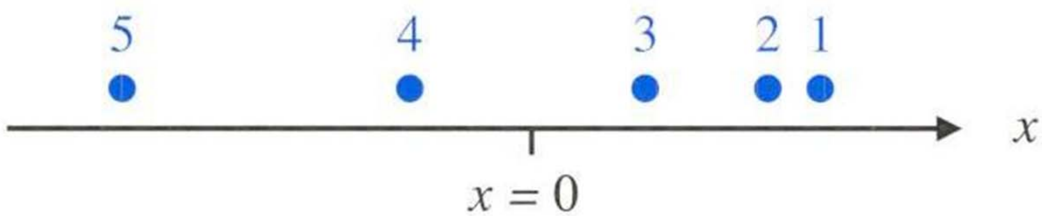
$$v_{av_x} = \frac{x - x_o}{t} \; ; \quad v_{av_x} = \frac{v_x + v_{o\,x}}{2}$$

$$(x - x_o) = \frac{v_x{}^2 - v_o{}^2{}_x}{2a}$$

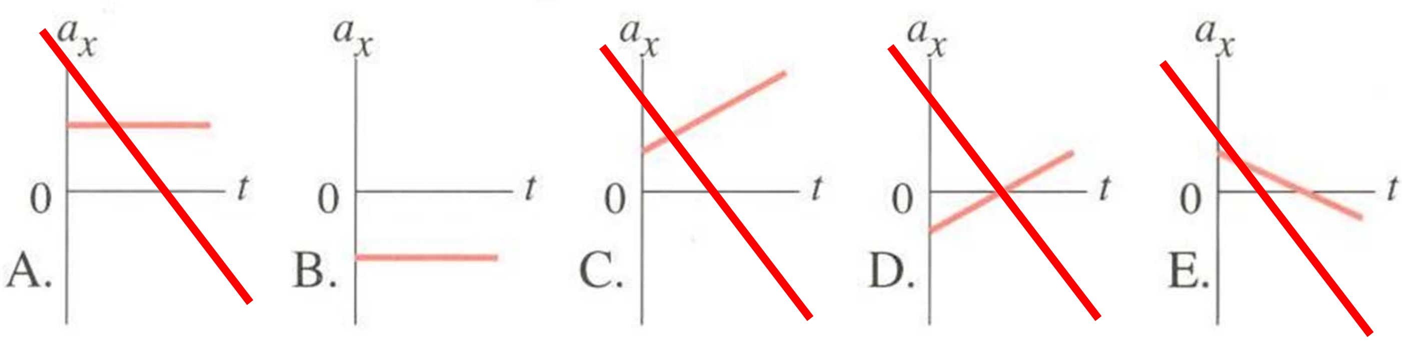
- Free Fall: replace $\begin{Bmatrix} x \\ a \end{Bmatrix}$ with $\begin{Bmatrix} y \\ -g \end{Bmatrix}$

i-Clicker!!

This is a motion diagram of an object moving along the x–direction with constant acceleration. The dots 1, 2, 3, … show the position of the object at equal time intervals Δt.



Which of the following a_x–t graphs best matches the motion shown in the motion diagram?



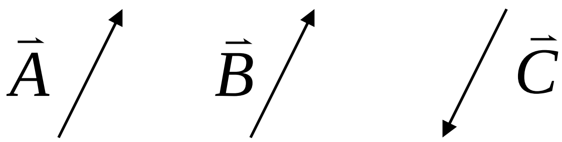
MOTION IN 2 AND 3 DIMENSIONS

Recall: Displacement is a vector !

- Has <u>magnitude</u> and <u>direction</u>

  So are velocity and acceleration

- Represent vector by arrow:



Here: $\vec{B} = \vec{A}$ (Same magnitude and direction)

Do <u>not</u> need to lie on top of each other

Furthermore: $\vec{C} = -\vec{A} \quad \left[= -\vec{B}\right]$

(Same Magnitude, Opposite Direction)

Notation: $|\vec{A}|$ = Magnitude of vector $\vec{A}$

Write: $|\vec{A}| = A$

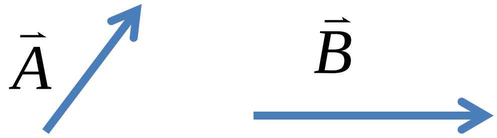
<u>NOTE</u>: For vectors above:

$|\vec{B}| = B = A$

and $|\vec{C}| = C = A$ **!** (- sign indicates direction)

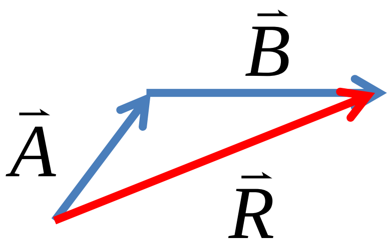
VECTOR ALGEBRA

Given:



What is $\vec{A}+\vec{B}$?

Consider displacement, first by $\vec{A}$ then by $\vec{B}$:

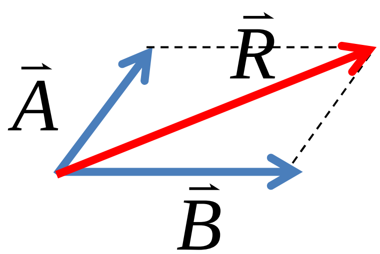


Net displacement is $\vec{R}$

→ $\vec{A}+\vec{B}=\vec{R}$

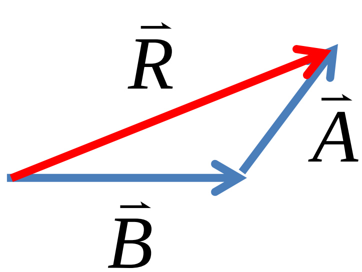
Geometrical addition by "head to tail" method.

- Consider parallelogram: $\vec{R}$ is the diagonal



$$\vec{R}=\vec{A}+\vec{B}$$

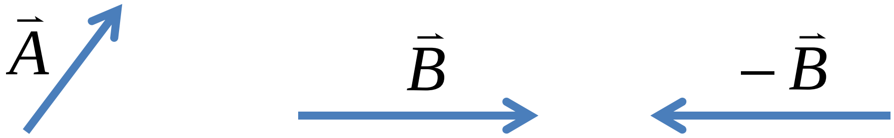
- But we can also do:

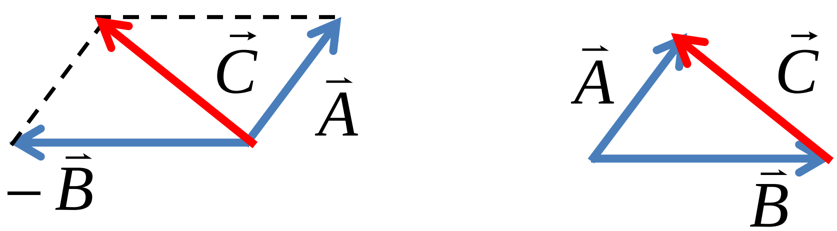


→ $\vec{R}=\vec{B}+\vec{A}$

Vector addition is commutative: $\vec{B}+\vec{A}=\vec{A}+\vec{B}$

- Subtraction: $\vec{A}-\vec{B}=?$ $\quad\quad \vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

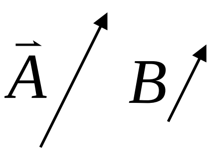




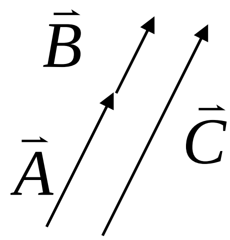
Notice: $\vec{C}=\vec{A}-\vec{B} \rightarrow \vec{B}+\vec{C}=\vec{A}$

$\vec{C}$ is vector from tip of $\vec{B}$ to tip of $\vec{A}$!

- Adding parallel vectors:



$$\vec{C}=\vec{A}+\vec{B}$$



If vectors parallel: $C = A + B$

If vectors anti parallel: $C = |A - B|$

In general: $|A - B| \leq C \leq A + B$

- Vector multiplied by scalar is a vector

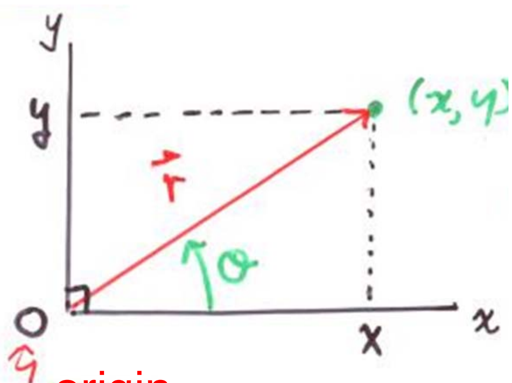
$\vec{C}=k\vec{A} \Rightarrow C = kA$ <u>Same direction</u>
<u>different magnitude</u>

VECTOR ANALYSIS

Any point in plane located by a vector

$\vec{r}$

from origin to point



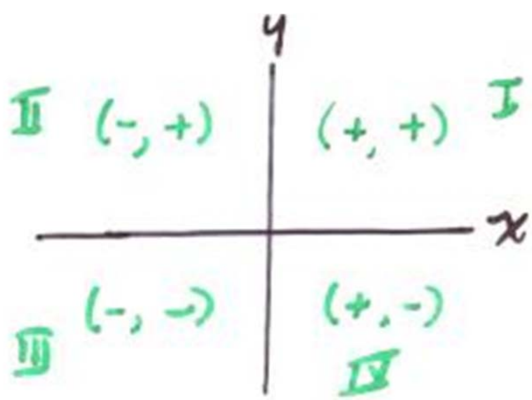
origin

θ always ccw from x-axis

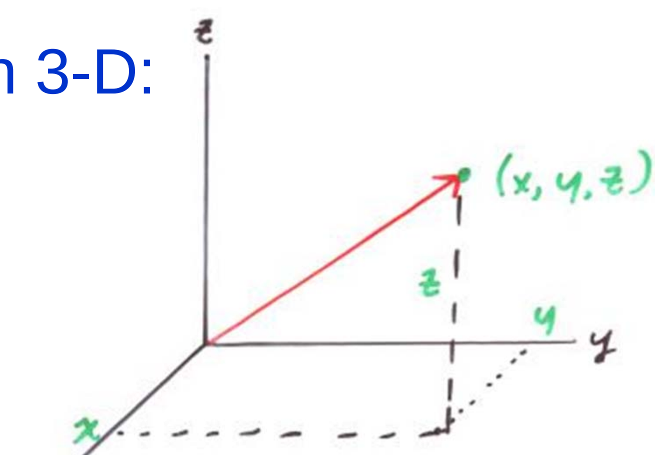
- In 2-D: (x, y) or (r, θ)

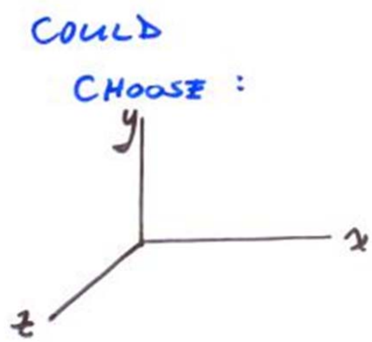
Cartesian Polar

Quadrants:



- In 3-D:





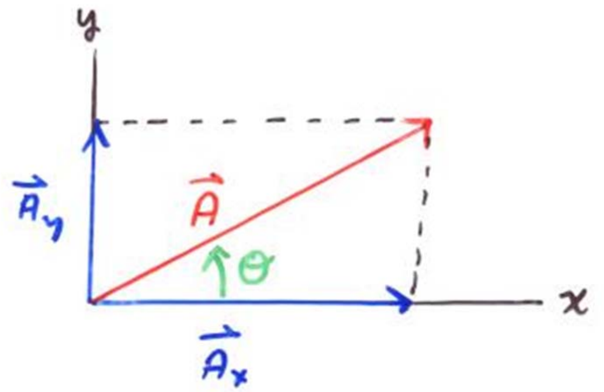
But: system must be right handed

$(x \rightarrow y \rightarrow z)$

COMPONENTS AND *COMPONENT* VECTORS

Clearly:

$$\vec{A} = \vec{A}_x + \vec{A}_y$$



$\vec{A}_x, \vec{A}_y$ are <u>component vectors</u> of $\vec{A}$

A_x, A_y are <u>components</u> of $\vec{A}$

Notice: $$\frac{A_y}{A} = \sin\theta \;\; => A_y = A\sin\theta$$

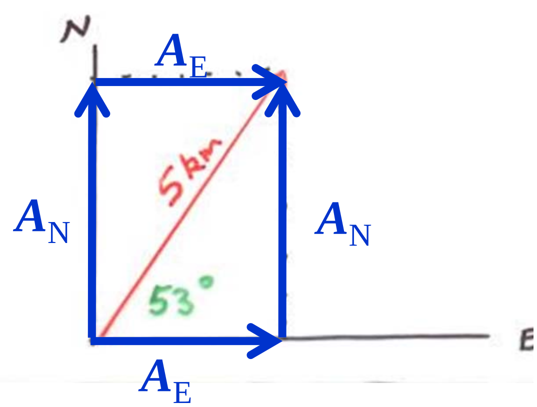
$$\frac{A_x}{A} = \cos\theta \;\; => A_x = A\cos\theta$$

<u>and</u> $$\frac{A_y}{A_x} = \frac{A\sin\theta}{A\cos\theta} = \tan\theta \;\; ; \;\; A_x^2 + A_y^2 = A^2$$

EXAMPLE: You are driving east on Canal Street. The Empire State building is 5km at 53º N of E. How many km E and then N must you drive?

$A_E = A\cos(53°) = (5\text{km})(0.6) = 3\text{ km}$

$A_N = A\sin(53°) = (5\text{ km})(0.8) = 4\text{ km}$



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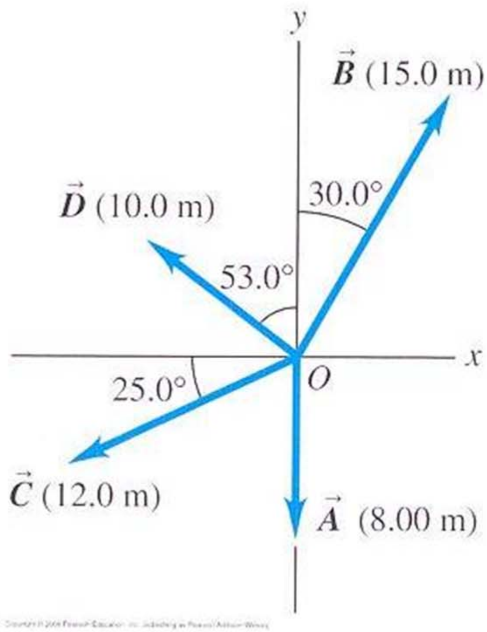


Consider the vectors shown. Which is a correct statement about $\vec{A} + \vec{B}$?

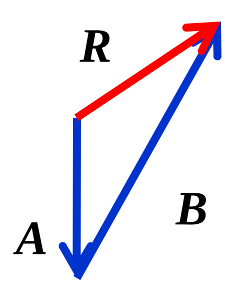
A. x–component > 0, y–component > 0

B. x–component > 0, y–component < 0

C. x–component < 0, y–component > 0

D. x–component < 0, y–component < 0

E. x–component = 0, y–component > 0

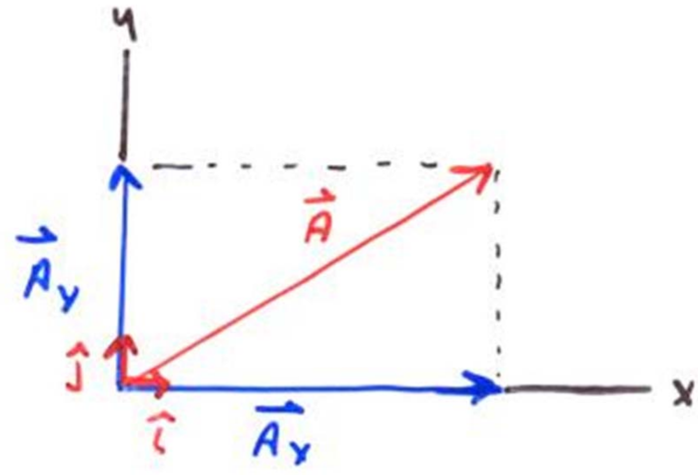


UNIT VECTORS

Consider vectors of unit length in
x- and y-directions

$$\hat{i}\ ;\ \ |\hat{i}| = 1$$

$$\hat{j}\ ;\ \ |\hat{j}| = 1$$

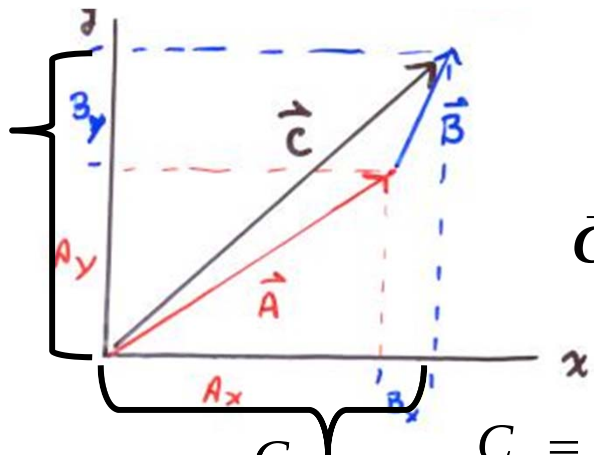


$$\vec{A}_x = A_x\hat{i}\ \ ;\ \ \vec{A}_y = A_y\hat{j}\ \ \Rightarrow\ \ \vec{A} = A_x\hat{i} + A_y\hat{j}$$

Unit vectors make vector addition very easy!

C_y

$$C_y = A_y + B_y$$



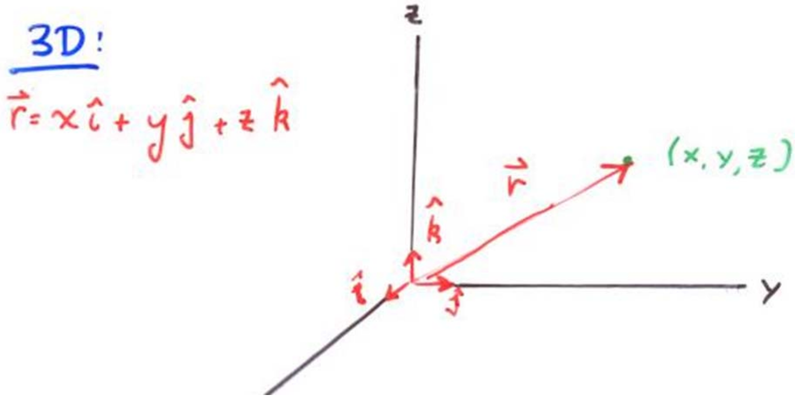
$$\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$$

$$\vec{\boldsymbol{C}} = C_x\hat{\boldsymbol{i}} + C_y\hat{\boldsymbol{j}}$$

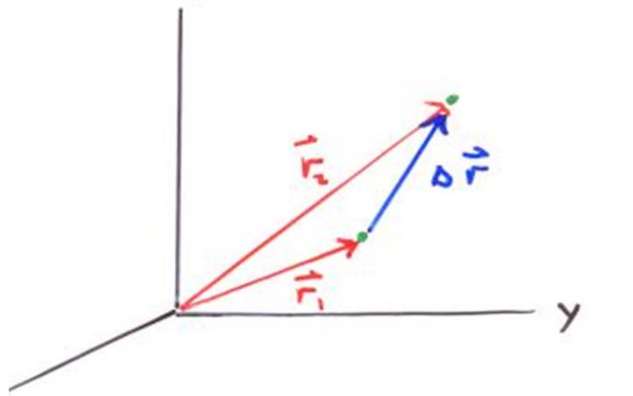
C_x

$$C_x = A_x + B_x$$

$$\vec{\boldsymbol{C}} = \vec{A} + \vec{B} = C_x\hat{\boldsymbol{i}} + C_y\hat{\boldsymbol{j}} = (A_x + B_x)\hat{\boldsymbol{i}} + (A_y + B_y)C_y\hat{\boldsymbol{j}}$$



DISPLACEMENT



$$\vec{r}_1 + \Delta\vec{r} = \vec{r}_2$$

$$\boxed{\Delta\vec{r} = \vec{r}_2 - \vec{r}_1}$$

$\Delta\vec{r}$ is displacement

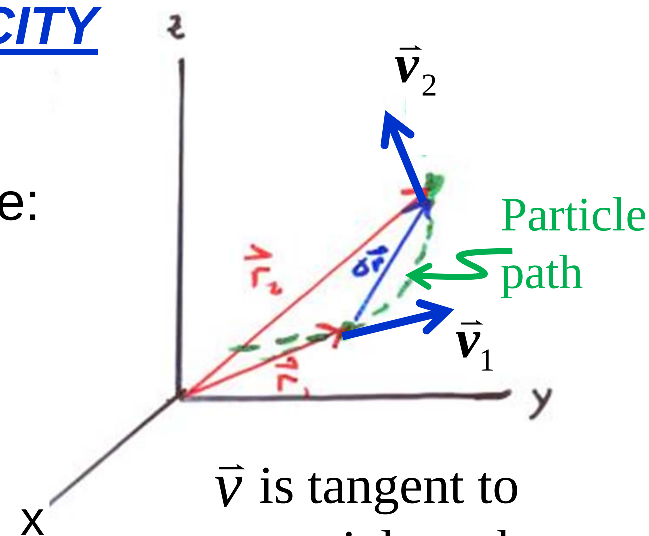
Suppose this displacement occurred in interval $(t_2 - t_1)$

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \vec{v}_{av}$$ Average velocity

INSTANTANEOUS VELOCITY

In analogy with 1-D case:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



$\vec{v}$ is tangent to particle path

$$\boxed{\vec{\boldsymbol{r}} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}}$$

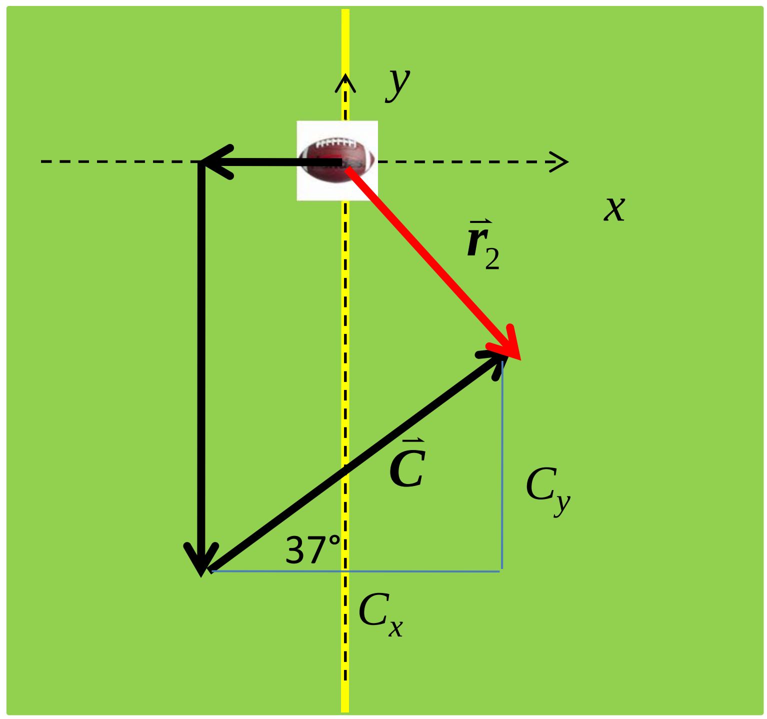
$$\boxed{\vec{\boldsymbol{v}} = \frac{dx}{dt}\hat{\boldsymbol{i}} + \frac{dy}{dt}\hat{\boldsymbol{j}} + \frac{dz}{dt}\hat{\boldsymbol{k}} = v_x\hat{\boldsymbol{i}} + v_y\hat{\boldsymbol{j}} + v_z\hat{\boldsymbol{k}}}$$

EXAMPLE: In one play during the RU football game last Saturday, the ball was snapped to the quarterback who dropped back 1 yd in 0.5 s, he then ran 5 yds parallel to the line of scrimmage in 1 s, then he threw the ball 5 yds at an angle of 37° from the forward direction in 0.5 s.
What was the average velocity, $\vec{\boldsymbol{v}}_{\text{av}}$, of the ball during this play?

$$\vec{\boldsymbol{v}}_{\text{av}} = \frac{\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1}{\text{t}_2 - \text{t}_1}$$

$$\vec{\boldsymbol{r}}_1 = 0 \qquad \vec{\boldsymbol{r}}_2 = ?$$

$$\vec{\boldsymbol{r}}_2 = (-1\text{ yd})\hat{\boldsymbol{i}} + (-5\text{ yd})\hat{\boldsymbol{j}} + \vec{\boldsymbol{C}}$$



$$\vec{\boldsymbol{r}}_2 = (-1\text{ yd})\hat{\boldsymbol{i}} + (-5\text{ yd})\hat{\boldsymbol{j}} + (4\text{ yd})\hat{\boldsymbol{i}} + (3\text{ yd})\hat{\boldsymbol{j}}$$

$C_x = C\cos(37°) = (5\text{ yds})(0.8) = 4\text{ yds}$

$C_y = C\sin(37°) = (5\text{ yds})(0.6) = 3\text{ yds}$

$$\vec{r}_2 = (3\text{ yd})\hat{\boldsymbol{i}} + (-2\text{ yd})\hat{\boldsymbol{j}}$$

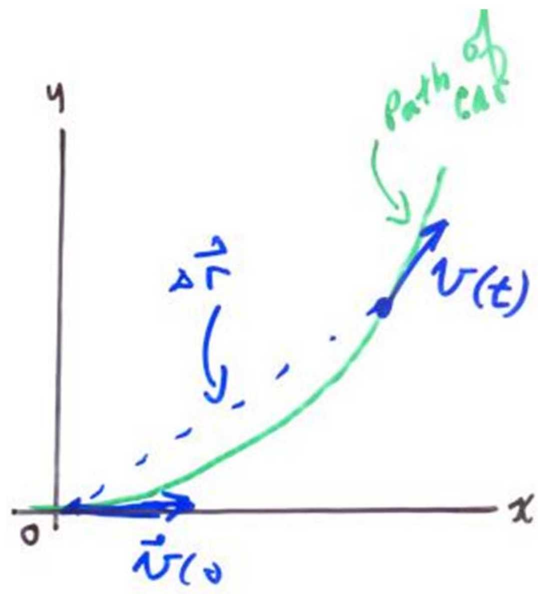
$\Delta t = 2\text{ s}$

$$\vec{v}_{av} = \frac{(3\text{ yd})\hat{\boldsymbol{i}} - (2\text{ yd})\hat{\boldsymbol{j}}}{2\text{ s}} = (1.5\,\frac{\text{yd}}{\text{s}})\hat{\boldsymbol{i}} - (1\,\frac{\text{yd}}{\text{s}})\hat{\boldsymbol{j}} = \vec{\boldsymbol{v}}_{av}$$

EXAMPLE: Suppose you drive on a curved road. Your position is given by: $x = v_o t + ct^2; \quad y = kt^3$

(Note: car at origin when t = 0)

- What is $v(t = 0)$?
- What is $v(t = 3\text{s})$?



$$\vec{r} = (v_o t + ct^2)\hat{i} + (kt^3)\hat{j}$$

$$\vec{v} = (v_o + 2ct)\hat{i} + (3kt^2)\hat{j}$$

$$\Rightarrow \vec{v}(t=0) = v_o \hat{i}$$

$$\vec{v}(t=3) = (v_o + 6c)\hat{i} + (27k)\hat{j}$$

- What is $\vec{v}_{av}$ for the interval 0 → 3s ?

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{\Delta t} \; ; \qquad \vec{r}(t=0) = 0$$

$$\vec{r}(t=3) = (3v_o + 9c)\hat{i} + (27k)\hat{j}$$

$$\Rightarrow \boxed{\vec{v}_{av} = (v_o + 3c)\hat{i} + (9k)\hat{j}}$$

i-Clicker!!

At a certain instant in time, the location airplane is given by:

$$\vec{r}_1 = 10\text{ km}\,\hat{\boldsymbol{i}} + 10\text{ km}\,\hat{\boldsymbol{j}} + 10\text{ km}\,\hat{\boldsymbol{k}}$$

One hour later, the plane is located at:

$$\vec{r}_2 = -10\text{ km}\,\hat{\boldsymbol{i}} + 20\text{ km}\,\hat{\boldsymbol{j}} + 10\text{ km}\,\hat{\boldsymbol{k}}$$

The average velocity of the plane during that hour is:

A: $\vec{v}_{av} = (0\,\hat{\boldsymbol{i}} + 30\,\hat{\boldsymbol{j}} + 20\,\hat{\boldsymbol{k}})\text{ km/hr}$

B: $\vec{v}_{av} = (0\,\hat{\boldsymbol{i}} - 30\,\hat{\boldsymbol{j}} - 20\,\hat{\boldsymbol{k}})\text{ km/hr}$

C: Cannot tell. Need to know position during that hour.

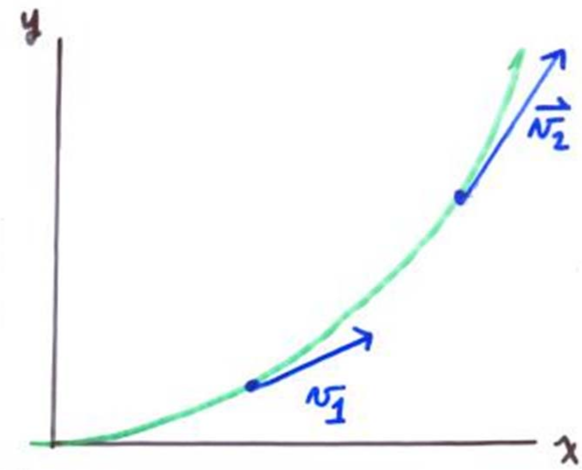
D: $\vec{v}_{av} = (-20\,\hat{\boldsymbol{i}} + 10\,\hat{\boldsymbol{j}} + 0\,\hat{\boldsymbol{k}})\text{ km/hr}$

E: $\vec{v}_{av} = (20\,\hat{\boldsymbol{i}} - 10\,\hat{\boldsymbol{j}} + 0\,\hat{\boldsymbol{k}})\text{ km/hr}$

$$\vec{v}_{\text{av}} = \frac{\vec{r}_2 - \vec{r}_1}{\text{t}_2 - \text{t}_1}$$

ACCELERATION VECTOR

Consider previous example:

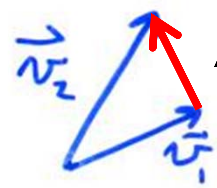


$\vec{v}$ changes direction

$\vec{v}$ changes magnitude

- Average acceleration:

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$



$$(\vec{a}_{av} \parallel \Delta\vec{v})$$

- Instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \to o} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$\vec{a} \neq 0$ for any curved path

$\vec{a} \neq 0$ if $\vec{v}$ changes: magnitude

or

direction

or

both

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2t}{dt^2}\hat{k}$$

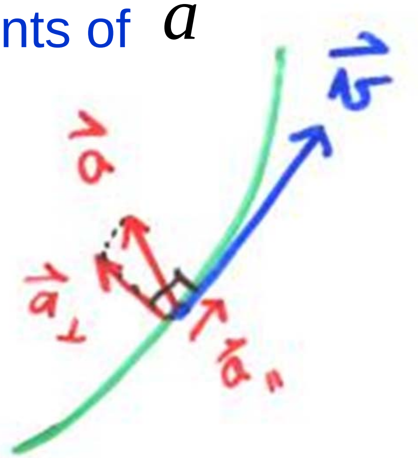
EXAMPLE: Consider car again:

$$\vec{r} = x\hat{i} + y\hat{j} = (v_o t + ct^2)\hat{i} + (kt^3)\hat{j}$$

$$\vec{\boldsymbol{v}} = \quad (v_o + 2ct)\hat{\boldsymbol{i}} + (3kt^2)\hat{\boldsymbol{j}}$$

$$\vec{a} = \quad (2c)\hat{\boldsymbol{i}} + (6kt)\hat{\boldsymbol{j}}$$

- Parallel and perpendicular components of $\vec{a}$



$\vec{a}_{\parallel}$: changes magnitude, v

$\vec{a}_{\perp}$: changes direction of $\vec{v}$

If: $a_{\parallel} = 0 \rightarrow$ $\vec{v}$ only changes direction

$\vec{a}_{\parallel}$ same direction as $\vec{v} \rightarrow$ v increases

$\vec{a}_{\parallel}$ opposite (anti parallel) to $\vec{v} \rightarrow$ v decreases