LECTURE 11 - MOMENTUM

Chapter 9
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TODAY’S OBJECTIVES

- Learn Center of Mass
- Collisions and Impulse
- Momentum
- Kinetic Energy in Collisions
- Elastic Collisions
CENTER OF MASS

**Center of mass**: point that moves as though
1.) all of the system’s mass were concentrated there
2.) all external forces were applied there
2 PARTICLE SYSTEM

Center of Mass (com)

\[ x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \]
MANY PARTICLES

\[ x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{M} \]

\[ = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]

where

\[ M = \sum_{i=1}^{n} m_i \]
THREE DIMENSIONS

\[ x_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]

\[ y_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \]

\[ z_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i \]

What does this look like?
THREE DIMENSIONS

\[ x_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]
\[ y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \]
\[ z_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i \]

Recall

\[ \vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k} \]

\[ \vec{r}_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \]
Consider the Earth and Mars in their orbit around the Sun. Where is the center of mass located for this three body system?

a) It is closer to the Earth, than it is to either the Sun or Mars.

b) It is at the center of a triangle that has the Sun at one apex, the Earth at another apex, and Mars at the third apex.

c) It is half of the distance between the Sun and Mars.

d) It is closer to the Sun, than it is to either the Earth or Mars.

e) It is closer to Mars, than it is to either the Earth or the Sun.
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e) It is closer to Mars, than it is to either the Earth or the Sun.
SOLID BODIES

Not everything can easily be separated into point like particles.

How do we calculate the center of mass of an ordinary object?
SOLID BODIES

\[ x_{com} = \frac{1}{M} \int x \, dm \quad \text{Recall the density} \quad \rho = \frac{M}{V} \]

For an infinitesimal small mass \( dm \) then \( \rho = \frac{dm}{dV} \)

\[ x_{com} = \frac{1}{V} \int x \, dV \]
SOLID BODIES 3D

\[ x_{com} = \frac{1}{V} \int x \, dV \]

\[ y_{com} = \frac{1}{V} \int y \, dV \]

\[ z_{com} = \frac{1}{V} \int z \, dV \]
EXAMPLE: FIND CENTER OF MASS
FIND CENTER OF MASS

\[ \vec{r}_{4kg} = 0\hat{i} + 0\hat{j} \]
\[ \vec{r}_{5kg} = 3\hat{i} + 2\hat{j} \]
\[ \vec{r}_{6kg} = 1\hat{i} + 3\hat{j} \]
FIND CENTER OF MASS

\[ M = \sum_{i=1}^{n} m_i \]

\[ \bar{x} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]

\[ \bar{y} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \]
FIND TOTAL MASS

\[ M = \sum_{i=1}^{n} m_i = m_1 + m_2 + m_3 \]

\[ M = 4kg + 5kg + 6kg \]

\[ M = 15kg \]
\[ x_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]

\[ \mathbf{r}_{4\text{kg}} = 0\hat{i} + 0\hat{j} \quad \mathbf{r}_{5\text{kg}} = 3\hat{i} + 2\hat{j} \]

\[ \mathbf{r}_{6\text{kg}} = 1\hat{i} + 3\hat{j} \]

\[ x_{\text{com}} = \frac{1}{15\text{kg}} (4\text{kg} \cdot 0\text{m} + 5\text{kg} \cdot 3\text{m} + 6\text{kg} \cdot 1\text{m}) \]

\[ x_{\text{com}} = 1.4\text{m} \]
\[ y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \]

\[ \vec{r}_{4kg} = 0\hat{i} + 0\hat{j} \quad \vec{r}_{5kg} = 3\hat{i} + 2\hat{j} \]

\[ \vec{r}_{6kg} = 1\hat{i} + 3\hat{j} \]

\[ y_{\text{com}} = \frac{1}{15kg} \left( 4kg \cdot 0m + 5kg \cdot 2m + 6kg \cdot 3m \right) \]

\[ y_{\text{com}} = 1.9m \]
\[ \vec{r}_{\text{com}} = 1.4 \hat{i} + 1.9 \hat{j} \]

\[ x_{\text{com}} = 1.4 \text{m} \quad y_{\text{com}} = 1.9 \text{m} \]
A high school physics teacher also happens to be the junior hockey team coach. During a break at practice, the coach asks two players to go to the center of the ice with a 10.0-m pole. A 40-kg player is at one end of the pole and a 60-kg player is at the other end. The players then start pulling themselves together by pulling the rod and sliding on the ice as they move along the rod. When the two players meet, what distance will the 60-kg player have moved?

a) zero m

b) 4.0 m

c) 5.0 m

d) 6.0 m

e) 10.0 m
A hockey coach asks two players to go to the center of the ice with a 10.0-m pole. A 40-kg player is at one end of the pole and a 60-kg player is at the other end. The players then start pulling themselves together by pulling the rod and sliding on the ice as they move along the rod. When the two players meet, what distance will the 60-kg player have moved?

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NEWTON’S 2ND LAW

\[ \vec{F}_{net} = M \vec{a}_{com} \]

System of particles

3Dimensions

\[ F_{net,x} = Ma_{com,x} \]
\[ F_{net,y} = Ma_{com,y} \]
\[ F_{net,z} = Ma_{com,z} \]
BILLIARD BALLS

https://www.youtube.com/watch?v=tlEVFgQEQQkU
BILLIARD BALLS

\[ v \rightarrow \quad \text{com} \quad \quad v = 0 \]

Assume no friction
Ignore rolling motion
Center of mass remains half way between the two balls (of identical mass)
BILLIARD BALLS

com
BILLIARD BALLS

\[ v = 0 \quad \text{com} \quad v \rightarrow \]
BILLIARD BALLS

We can follow the velocity of the com, which remains constant.

\[ \nu \rightarrow \text{com} \]

Assume no friction
Ignore rolling motion
BILLIARD BALLS

We can follow the velocity of the com, which remains constant.

Center of mass remains half way between the two balls (of identical mass)
BILLIARD BALLS

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\[ v \rightarrow \]
BILLIARD BALLS

We can follow the velocity of the com, which remains constant
WHAT IS THE NET FORCE ON THE BILLIARD BALL SYSTEM?

- **A.** \[ F_{net} = m \left( \frac{dv_1}{dt} + \frac{dv_2}{dt} \right) \]
- **B.** \[ F_{net} = m \frac{v^2}{R} \]
- **C.** \[ F_{net} = m \frac{dv_{com}}{dt} \]
- **D.** \[ F_{net} = m \left( \frac{dv_1}{dt} - \frac{dv_2}{dt} \right) \]
- **E.** \[ F_{net} = 0 \]
WHAT IS THE NET FORCE ON THE BILLIARD BALL SYSTEM?

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- B.) \[ F_{net} = m \frac{v^2}{R} \]
- C.) \[ F_{net} = m \frac{dv_{com}}{dt} \]
- D.) \[ F_{net} = m \left( \frac{dv_1}{dt} - \frac{dv_2}{dt} \right) \]
- E.) \[ F_{net} = 0 \]
BILLIARD BALLS

Since the $v_{com} = const$ remains constant, there is no external forces on the system.

At the point of collision, the forces are *internal*
$F_{\text{net,com}}$

- Only include external forces
- For a closed system, $M$ is the total mass
- $\vec{a}_{\text{com}}$ is the acceleration of the center of mass, no information is given about the individual components of the system
BASEBALL BAT THROWN

Center of mass follows typical path of projectile motion
Center of mass follows typical path of projectile motion
Suppose each piece of the exploded firework has a force $\vec{F}_i$.

The original mass of the firework was $M$ and it breaks into 1000 pieces, what is the acceleration?

\[ \vec{F}_{net} = M \vec{a}_{com} \]

\[ \sum_{i=1}^{1000} \vec{F}_i = M \vec{a}_{com} \]

\[ \vec{a}_{com} = \frac{1}{M} \sum_{i=1}^{1000} \vec{F}_i \]
LINEAR MOMENTUM

Linear momentum \[ \vec{p} = m \vec{v} \]

Units \[ kg \cdot \frac{m}{s} \]

How does this relate to force?
MOMENTUM AND FORCE

\[ \vec{F}_{net} = \frac{d\vec{p}}{dt} \]

Assume mass is constant, then

\[ \vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d}{dt} (m\vec{v}) = m\vec{a} \]

If no external forces, \( \vec{p} = \text{const} \)
SYSTEM OF PARTICLES

Linear momentum

\[ \vec{P} = M \vec{v}_{com} \]
\[ = p_1 + p_2 + \ldots + p_3 \]
\[ = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_3 \vec{v}_3 \]

Force

\[ \vec{F}_{net} = \frac{d\vec{P}}{dt} \]
A 9-kg object is at rest. Suddenly, it explodes and breaks into two pieces. The mass of one piece is 6 kg and the other is a 3-kg piece. Which one of the following statements concerning these two pieces is correct?

a) The speed of the 6-kg piece will be one eighth that of the 3-kg piece.
b) The speed of the 3-kg piece will be one fourth that of the 6-kg piece.
c) The speed of the 6-kg piece will be one forth that of the 3-kg piece.
d) The speed of the 3-kg piece will be one half that of the 6-kg piece.
e) The speed of the 6-kg piece will be one half that of the 3-kg piece.
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e) The speed of the 6-kg piece will be one half that of the 3-kg piece.
COLLISIONS (CLOSED, ISOLATED SYSTEM)

Conservation of linear momentum
\[ \vec{P} = \text{const} \]

If no external forces,
\[ \vec{P} = \text{const} \]

Law of conservation of linear momentum
\[ \vec{P}_i = \vec{P}_f \]

But these are vectors, so if the net external force along an axis is zero, then the components of linear momentum cannot change
\[ F_{\text{net},x} = 0 \quad \Rightarrow \quad p_{x,i} = p_{x,f} \]
TYPES OF COLLISIONS

- **Elastic collisions** - total kinetic energy and momentum are conserved

- **Inelastic collisions** - linear momentum is conserved

- **Completely inelastic collisions** - linear momentum is conserved and the colliding bodies stick together afterwards

Demo Newton’s cradle
ELASTIC COLLISIONS

\[ \vec{P}_{\text{net},i} = \vec{P}_{\text{net},f} \quad K_i = K_f \]

Stationary Target

\[ v_{1,i} \quad v_{2,i} = 0 \]

\[ m_1 v_{1,i} = m_1 v_{1,f} + m_2 v_{2,f} \]

\[ \frac{1}{2} m_1 v_{1,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 \]
ELASTIC COLLISIONS

Stationary Target

\[ \frac{1}{2} m_1 v_{1,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 \]

\[ m_1 v_{1,i}^2 = m_1 v_{1,f}^2 + m_2 v_{2,f}^2 \]

\[ m_1 \left( v_{1,i}^2 - v_{1,f}^2 \right) = m_2 v_{2,f}^2 \]

\[ m_1 \left( v_{1,i} - v_{1,f} \right) \left( v_{1,i} + v_{1,f} \right) = m_2 v_{2,f}^2 \]

\[ v_{2,i} = 0 \]
ELASTIC COLLISIONS

Stationary Target

After some algebra

\[ v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i} \quad v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,i} \]
ELASTIC COLLISIONS (STATIONARY TARGET) SPECIAL CASES

Equal Masses

\[ v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i} \]
\[ v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,i} \]
\[ v_{1,f} = \frac{m - m}{m + m} v_{1,i} = 0 \]
\[ v_{2,f} = \frac{2m}{m + m} v_{1,i} = v_{1,i} \]

The second particle has the same velocity as \( v_{1,i} \) after the collision.
ELASTIC COLLISIONS (STATIONARY TARGET) SPECIAL CASES

Massive Target

\[ m_2 \gg m_1 \]

\[ v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i} \]

\[ v_{1,f} \approx -\frac{m_2}{m_2} v_{1,i} \]

\[ v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,i} \]

\[ v_{2,f} \approx \frac{2m_1}{m_2} v_{1,i} \]

The first particle bounces back, target moves forward at low speed.
ELASTIC COLLISIONS (STATIONARY TARGET) SPECIAL CASES

**Massive Projectile**

\[ m_2 < < m_1 \]

\[ v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i} \]

\[ v_{1,f} \approx \frac{m_1}{m_1} v_{1,i} \]

\[ v_{1,f} \approx v_{1,i} \]

\[ v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,i} \]

\[ v_{2,f} \approx \frac{2m_1}{m_1} v_{1,i} \]

\[ v_{2,f} \approx 2v_{1,i} \]

First particle continues, second goes ahead at twice the speed
ELASTIC COLLISIONS

Both moving

\[
\frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2
\]

\[
m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}
\]
ELASTIC COLLISIONS
(DEMO CARTS)

Both moving

\[ v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i} + \frac{2m_2}{m_1 + m_2} v_{2,i} \]

\[ v_{1,f} = \frac{2m_1}{m_1 + m_2} v_{1,i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2,i} \]
COMPLETELY INELASTIC COLLISIONS
COMPLETELY INELASTIC COLLISIONS

If \( m_2 \) is initially at rest

\[
m_1 v_1 = (m_1 + m_2) V
\]

Only linear momentum is conserved

\[
V = \frac{m_1}{m_1 + m_2} v_{1,i}
\]
The center of mass continues at the same velocity even after the collision.

\[ \vec{v}_{\text{com}} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2} \]
INELASTIC COLLISIONS IN 1D

\[ \vec{p}_{1,i} + \vec{p}_{2,i} = \vec{p}_{1,f} + \vec{p}_{2,f} \]

\[ m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f} \]

Note, unlike for elastic collisions, the kinetic energy is NOT conserved so we cannot use that as an additional constraint here.
Darrin, $m = 45 \text{ kg}$, runs and jumps off a stationary, 168-kg floating platform on a lake. Darrin’s velocity as he leaps is $+2.7 \text{ m/s}$. Ignoring any frictional effects, what is the recoil velocity of the platform?

a) $-2.7 \text{ m/s}$

b) $+0.72 \text{ m/s}$

c) $-1.4 \text{ m/s}$

d) $-0.72 \text{ m/s}$

e) $+2.7 \text{ m/s}$
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a) −2.7 m/s

b) +0.72 m/s

c) −1.4 m/s

d) −0.72 m/s

e) +2.7 m/s
Consider the following objects:
(1) an electron \( (m = 9.1 \times 10^{-31} \text{ kg}, \, v = 5.0 \times 10^7 \text{ m/s}) \)
(2) the Hubble Space Telescope \( (m = 1.1 \times 10^4 \text{ kg}, \, v = 7.6 \times 10^3 \text{ m/s}) \)
(3) a snail \( (m = 0.02 \text{ kg}, \, v = 0.0003 \text{ m/s}) \)
(4) the largest super oil tanker \( (m = 1.5 \times 10^8 \text{ kg}, \, v = 2.0 \text{ m/s}) \)
(5) a falling rain drop \( (m = 0.0002 \text{ kg}, \, v = 9.5 \text{ m/s}) \)

Which one of these objects requires the greatest change in momentum to stop moving?

a) 1
b) 2
c) 3
d) 4
e) 5
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(1) an electron \((m = 9.1 \times 10^{-31} \text{ kg}, \ v = 5.0 \times 10^7 \text{ m/s})\)

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a) 1

b) 2

c) 3

da) 4

e) 5
Recall that a change in momentum only occurs when a force acts on an object (e.g. collision)

\[ \vec{F} = \frac{d\vec{p}}{dt} \]

A change in momentum indicates the magnitude and duration of a collisional force

Impulse \[ \Delta \vec{p} = \vec{J} \]

Linear momentum-impulse theorem
Recall that a change in momentum only occurs when a force acts on an object (e.g. collision)

\[ \vec{F} = \frac{dp}{dt} \]

We can also talk about average impulse where

\[ \text{Impulse} \quad J = F_{\text{avg}} \Delta t \]
IMPULSE- AFFECTS “FEEL” OF A COLLISION
**IMPULSE - REDUCE IMPACT**

\[
\text{Impulse} = F_{\text{average}} \Delta t = m \Delta v
\]

- Reduce average impact force
- Extend time of collision
- For a given change in momentum, the impulse stays constant.
NEXT WEEK

- Gravitation