THE END IS NEAR
GAVITATION

REVIEW

• Center of Mass:

\[ x_{cm} = \frac{\sum m_i x_i}{M} \quad r_{cm} = \frac{\sum m_i r_i}{M} \]

• Motion of CM

\[ \sum_i F_{ext} = Ma_{cm} = \frac{dP}{dt} \]

• Newton’s Law of Gravitation:

\[ F_1 = F_2 = G \frac{m_1 m_2}{r^2} \]

Note that CM is not the same as the center of gravity! We cannot, in general, replace an extended mass by a point mass at the CM and expect the same net gravitation force to act. (Exception: If the gravitational acceleration is the same everywhere.)
A yellow block and a red rod are joined together. Each object is of uniform density. The center of mass of the combined object is at the position shown by the black “X.”

Which has the greater mass, the yellow block or the red rod?

A. the yellow block  
B. the red rod  
C. they both have the same mass  
D. not enough information given to decide

Choose origin \( x = 0 \) at CM (black X):

\[
m_1 x_{cm_1} + m_2 x_{cm_2} = 0
\]

\[
m_1 |x_{cm_1}| = m_2 |x_{cm_2}|
\]
Gravitational Potential Energy

Recall: Near the surface of the Earth, the force of gravity can be considered approximately constant (F=-mg). The work done by gravity is

\[ W = \int_{h_1}^{h_2} F(h)dh \]

\[ = -mg(h_2 - h_1) \]

Define: gravitational potential energy \( U = mgh. \)

In terms of \( U \), the work can be written as \( W = U_1 - U_2. \)

Now let's generalize this, accounting for the weakening of gravity with distance from the Earth:

\[ W = \int_{r_1}^{r_2} F(r)dr \]

\[ = -GmM \int_{r_1}^{r_2} \frac{1}{r^2}dr \]

\[ = -GmM \left( \frac{1}{r_1} - \frac{1}{r_2} \right). \]

We can again write this as \( W = U_1 - U_2 \)

if the gravitational potential energy is

\[ U = -G \frac{Mm}{r} \]

The zero point (\( U = 0 \)) is arbitrarily chosen at infinity (\( r = \infty \)).
Gravitational potential energy

\[ U_{\text{gravity}}(r) = - \int_{r_{\text{ref}}}^{r} \mathbf{F} \cdot d\mathbf{r} \]

\[ \mathbf{F} = - \frac{Gm_1 m_2 \hat{r}}{r^2} \]

\[ U_{\text{gravity}}(r) = - \int_{\infty}^{r} \frac{-Gm_1 m_2}{r'^2} \, dr' = - \frac{Gm_1 m_2}{r} \]

**Example:**
What is the gravitational potential of a satellite at height \( h \)?

\[ U_{\text{gravity}}(r = R_E + h) = - \frac{G M_E m_S}{R_E + h} \]
We previously said the gravitational potential of an object of mass \( m \) at a height \( h \) is \( U = mgh \). How is this related to

\[
U_{gravity} = -\frac{GM_Em_s}{R_E + h} \quad ?
\]

A. No relation  
B. They are approximately equal  
C. They are exactly equal  
D. First equation is wrong  
E. Second equation is wrong

\[
\frac{1}{R_E + h} = \frac{1}{R_E} \frac{1}{1 + h/R_E} \approx \frac{1}{R_E} \left( 1 - h/R_E + \right) = \frac{1}{R_E} - \frac{h}{R_E^2} +
\]

\[
U_{gravity}(R_E + h) - U_{gravity}(R_E) = -\frac{GM_Em_m}{R_E + h} - \left( -\frac{GM_Em_m}{R_E} \right) \approx \frac{GM_E}{R_E^2} mh
\]
Gravitational Potential

- The **gravitational potential energy** is proportional to the test mass $m$ that we put at a given point.
- It makes sense to define the **gravitational potential** of the Earth as the **potential energy per unit mass**:

$$V(r) = \frac{U(r)}{m} = -G \frac{M}{r}$$

$r$: distance from center of Earth

This is an example of a **scalar field**: A scalar quantity $V(r)$ that is defined at every point $r$ in space.
Adding Potentials

Adding potentials for two masses involves a simple calculation.

\[
\text{The total potential at } P = -\frac{GM_A}{r_A} + -\frac{GM_B}{r_B}
\]

The potentials of several masses (Earth, Moon) add up:
Gravitational Field

We can do the same thing for the force: Divide the force by the test mass $m$, to arrive at a quantity that is defined at every point in space:

$$g(r) = \frac{F(r)}{m} = -G \frac{M}{r^2} \hat{r}$$

The vector $g$ is the acceleration of gravity. For an object at any given point in space $r$, $g(r)$ tells us the magnitude and direction of the object’s acceleration due to gravity.

This is an example of a vector field: A vector quantity $g(r)$ that is defined at every point $r$ in space. (Other important vector fields you will encounter are the electric and magnetic fields.)

For several masses (e.g., Earth and Moon), the gravitational field vectors (acceleration vectors) add up:
Field Lines

Relationship between field lines and potential

This picture has field lines and lines of equipotential.

Remember they are always perpendicular to each other (at right angles).

The lines of equipotential are closest where the field is strongest.

The field is strongest where the lines of equipotential are most dense.
A planet X has one half the radius of the Earth and one third its mass. **Find the acceleration of gravity at the surface of planet X in terms of the acceleration of gravity at the surface of the earth, \( g_{\text{Earth}} \).**

A. \( g_X = \frac{2}{3} g_{\text{Earth}} \)

B. \( g_X = \frac{4}{3} g_{\text{Earth}} \)

C. \( g_X = \frac{1}{6} g_{\text{Earth}} \)

D. \( g_X = \frac{1}{12} g_{\text{Earth}} \)

E. \( g_X = g_{\text{Earth}} \)

\[
g_X = \frac{GM_X}{R_X^2} = \frac{GM_{\text{E}}/3}{(R_{\text{E}}/2)^2} = \frac{4}{3} \frac{GM_{\text{E}}}{R_{\text{E}}^2} = \frac{4}{3} g_{\text{E}}
\]
A satellite is moving in a **circular orbit**.

In this case, the centripetal acceleration is provided by the gravitational force:

\[ \frac{mv^2}{r} = \frac{GM_Em}{r^2} \]

Solving for \( v \):

\[ v(r) = \sqrt{\frac{GM_E}{r}} = \frac{2\pi r}{T} \]
Elliptical Orbits

“Kepler’s Laws”
Escape Speed

- An object is shot away from Earth. How fast should it be moving to not fall back?
- Consider conservation of Energy:
  - $KE_{(initial)} + U_{(initial)} = KE_{(final)} + U_{(final)}$
  - $KE_{(initial)} + U_{(initial)} = 0$

\[
\frac{1}{2}mv^2 = G \frac{Mm}{r_E} \quad \Rightarrow \quad v_{escape} = \sqrt{\frac{2GM}{r_E}}
\]

On Earth this is 11 km/s

This is why the Moon has lost its atmosphere. The escape velocity for a molecule was similar to the speed of the molecules in its early atmosphere.
Spherical Shells of Mass

• The gravitational field outside a spherical shell is the same as if the mass were at the center.
• Same for spheres, such as stars and planets.
• The gravitational field inside a spherical shell is zero!
• The gravitational potential inside a spherical shell is constant:

![Diagram of gravitational field and potential inside and outside a spherical shell]
Iclicker:

In the orbiting space station:

A. Gravity is zero because of the great distance from Earth.
B. Gravity is zero because the space station is outside of the Earth’s atmosphere.
C. Gravity is not zero. Astronauts “fall” (accelerate) toward Earth, but the spacecraft accelerates at the same rate, creating the illusion of weightlessness.
D. Gravity is not actually zero. Being in outer space plays tricks on people’s minds.
E. Gravity is clearly zero, meaning Newton was wrong. Experiment beats theory every time.
Weightlessness

- Apparent weightlessness can occur when in free fall:
  - Consider when Stephen Hawking and the “Vomit Comet” both fall with an acceleration of g
Artificial Gravity

- In space it is relatively easy to simulate gravity by using a rotating ship.
- This “artificial gravity” has nothing to do with gravity, however. It just feels like gravity.
- Watch the [video](#).
- This would be very useful for a manned mission to Mars.
- Why is this idea not used on the Space Station?
Gravitational Lensing

1. A Distant Source
   Light leaves a young, star-forming blue galaxy near the edge of the visible universe.

2. A Lens Of 'Dark Matter'
   Some of the light passes through a large cluster of galaxies and surrounding dark matter, directly in the line of sight between Earth and the distant galaxy. The dark matter's gravity acts like a lens, bending the incoming light.

3. Focal Point: Earth
   Most of this light is scattered, but some is focused and directed toward Earth. Observers see multiple, distorted images of the background galaxy.
Black Holes

- **Black holes** are massive objects where gravity is so strong that the escape speed is greater than the speed of light.
- Nothing, not even light can escape.
- This is a simulated image of what a black hole would look like.
- Notice the *gravitational lensing* effect.
- To properly describe black holes, gravitational lensing, and other effects such as *gravitational waves*, we need to go beyond Newton’s theory:
- This is the domain of **Einstein’s General Relativity**.