Physics 115

Lecture: Rotational Kinematics
2016
Announcements

• HW quiz at beginning of Monday recitations
• Wednesday 2/3-Thursday 2/4
  Recitations will meet on the third floor of ARC
  in room 336
Angular measure: the radian

Half a radian would subtend an arc length equal to half the radius and 2 radians would subtend an arc length equal to two times the radius.

A general Radian Angle ($\Delta \theta$) subtends an arc length ($\Delta s$) equal to $R$. The theta in this case represents ANGULAR DISPLACEMENT.

$$\Delta s = R\Delta \theta$$
Angular Velocity

Since velocity is defined as the rate of change of position. **ANGULAR VELOCITY** is defined as the rate of change of **ANGULAR POSITION**.

\[
\bar{v} = \frac{\Delta x}{\Delta t} \rightarrow \text{translational velocity}
\]

\[
\bar{\omega} = \frac{\Delta \theta}{\Delta t} \rightarrow \text{rotational velocity}
\]

\[
v = \frac{dx}{dt}, \omega = \frac{d\theta}{dt}
\]

1 revolution = \(2\pi\) radians = 360°
Angular Acceleration

Once again, following the same lines of logic. Since acceleration is defined as the rate of change of velocity. We can say the ANGULAR ACCELERATION is defined as the rate of change of the angular velocity.

\[
\bar{a} = \frac{\Delta v}{\Delta t} \rightarrow \text{translational acceleration}
\]

\[
\bar{\alpha} = \frac{\Delta \omega}{\Delta t} \rightarrow \text{rotational acceleration}
\]

\[a = \frac{dv}{dt}, \alpha = \frac{d\omega}{dt}\]
A disc with a moment of inertia of 0.2 kg·m² rotates at 300 revolutions per minute. It takes 40 s for the disc to reach this rotation rate starting from rest. Consider a point A on the disc that is 1 cm from the axis of rotation and another point B that is farther from this axis.

Ten seconds after starting from rest:

(1) Will the magnitude of the angular acceleration of point A be greater than, less than, or equal to the magnitude of the angular acceleration of point B?

Answer: Less than since the linear velocity is the product of the angular velocity and the distance of the point from the axis of rotation and A is closer to the axis of rotation.
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(1) Will the magnitude of the angular acceleration of point A be greater than, less than, or equal to the magnitude of the angular acceleration of point B?

(2) Will the magnitude of the angular velocity of point A be greater than, less than, or equal to the magnitude of the angular velocity of point B?

(3) Will the magnitude of the linear velocity of point A be greater than, less than, or equal to the magnitude of the linear velocity of point B?

Answer: Less than since the linear velocity is the product of the angular velocity and the distance of the point from the axis of rotation and A is closer to the axis of rotation.
Angular and tangential quantities

- $\Delta x = \Delta \theta \ r$
- $v = \omega \ r$
- $a = \alpha \ r$
\[ x \leftrightarrow \theta \quad v_x \leftrightarrow \omega \quad a_x \leftrightarrow \alpha \]

**TRANSLATION**

\[
\begin{align*}
    v_x &= v_{x0} + a_x t \\
    x &= x_0 + v_{x0} t + \frac{1}{2} a_x t^2 \\
    v_x^2 &= v_{x0}^2 + 2a_x \Delta x
\end{align*}
\]

**ROTATION**

\[
\begin{align*}
    \omega &= \omega_0 + \alpha t \\
    \theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
    \omega^2 &= \omega_0^2 + 2\alpha \Delta \theta
\end{align*}
\]
A turntable speeds up at a rate of 12.0 rad/s² and turns through 400 radians in 6.00 seconds. How fast was it rotating when it started to speed up?
A turntable speeds up at a rate of $12.0 \text{ rad/s}^2$ and turns through 400 radians in 6.00 seconds. How fast was it rotating when it started to speed up?

What do we know?

$\alpha = 12 \text{ rad/s}^2$

$\Delta \theta = 400 \text{ rad}$

$t = 6 \text{ s}$

What do we want to know?

$\omega_o = ?$
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What do we know?

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\[ \Delta \theta = 400 \text{ rad} \]
\[ t = 6 \text{ s} \]

What do we want to know?

\[ \omega_o = ? \]

How is the unknown quantity related to the known quantities?

\[ \Delta \theta = \omega_o t + \frac{1}{2} \alpha t^2 \]
\[ 400 = \omega_o (6) + (0.5)(12)(6)^2 \]
\[ \omega_o = \]
A turntable speeds up at a rate of 12.0 rad/s$^2$ and turns through 400 radians in 6.00 seconds. How fast was it rotating when it started to speed up?

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$\alpha = 12 \text{ rad} / \text{s}^2$

$\Delta \theta = 400 \text{ rad}$

$t = 6 \text{ s}$

What do we want to know?

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How is the unknown quantity related to the known quantities?

$\Delta \theta = \omega_o t + \frac{1}{2} \alpha t^2$

$400 = \omega_o (6) + (0.5)(12)(6)^2$

$\omega_o = 30.7 \text{ rad/s}$
Annotated solution

A 900 kg car (including the tires) accelerates uniformly from rest and reaches a speed of 20 m/s in 10 sec. The diameter and mass of each tire is 58 cm and 5 kg, respectively. Find the final rotational speed of the tires in revolutions per second.
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\[ v_o = 0, \quad v_f = 20 \text{ m/s}, \quad t = 10 \text{ s}, \quad r = 29 \text{ cm} \]
Annotated solution

A 900 kg car (including the tires) accelerates uniformly from rest and reaches a speed of 20 m/s in 10 sec. The diameter and mass of each tire is 58 cm and 5 kg, respectively. Find the final rotational speed of the tires in revolutions per second.

What do we know? We know the initial speed, the final speed and the time it takes to attain that speed.; we also know the radius of the tires: \(v_o=0, \ v_f=20 \text{ m/s, } t=10\text{s, } r=29 \text{ cm}\)

What do we want to know?

\(\omega_f\)
Annotated solution

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\[ v_o = 0, \ v_f = 20 \ m/s, \ t = 10s, \ r = 29 \ cm \]

What do we want to know?

\( \omega_f \)

Since we know the translational speed of the car, we also know the translational speed of the tires. The final angular speed of the tires, \( \omega_f \), is equal to the \( v_{\text{translation}} / r_{\text{tires}} \). And to express the angular speed as a rotational speed, divide by \( 2\pi \).
Annotated solution

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What do we want to know?

\[ \omega_f \]

Since we know the translational speed of the car, we also know the translational speed of the tires. The final angular speed of the tires, \( \omega_f \), is equal to the \( \frac{v_{\text{translation}}}{r_{\text{tires}}} \). And to express the angular speed as a rotational speed, divide by \( 2\pi \).

\[ \omega_f = \left( \frac{v_{\text{translation}}}{r_{\text{tires}}} \right) \left( \frac{1}{2\pi} \right) = \left( \frac{20 \text{ m/s}}{0.29 \text{ m}} \right) \left( \frac{1 \text{ rev}}{2\pi \text{ radians}} \right) = 11 \text{ revolutions/s} \]
Moment of inertia

- Inertial quantity associated with rotational motion.
- Sum of the products of mass of the “masslet” and the (distance to the axis of rotation)$^2$
- For a discrete mass distribution: $I = \Sigma m_i r_i^2$
- For a continuous mass distribution:

\[ I = \int r^2 \, dm \]
Rotational Kinetic Energy

• Translational: $K_{\text{trans}} = \frac{1}{2}mv^2$
• Rotational: $K_{\text{rot}} = \frac{1}{2}I\omega^2$
A turntable speeds up at a rate of $12.0 \ \text{rad/s}^2$ and turns through 400 radians in 6.00 seconds. How fast was it rotating when it started to speed up?

**How much work was done to speed up the turntable?**

What do we know?

- $\alpha = 12 \ \text{rad/ s}^2$
- $\Delta \theta = 400 \ \text{rad}$
- $t = 6 \ \text{s}$

What do we want to know?

- $\omega_o = ?$

How is the unknown quantity related to the known quantities?

$$\Delta \theta = \omega_o t + \frac{1}{2} \alpha t^2$$

$$400 = \omega_o (6) + (0.5)(12)(6)^2$$

$$\omega_o = 30.7 \ \text{rad/s}$$
A turntable speeds up at a rate of 12.0 rad/s² and turns through 400 radians in 6.00 seconds. How fast was it rotating when it started to speed up?

How much work was done to speed up the turntable?

What do we know?

\[ \alpha = 12 \text{ rad} / \text{s}^2 \]
\[ \Delta \theta = 400 \text{ rad} \]
\[ t = 6 \text{ s} \]

What do we want to know?

\[ \omega_o = ? \]

How is the unknown quantity related to the known quantities?

\[ \Delta \theta = \omega_o t + \frac{1}{2} \alpha t^2 \]

\[ 400 = \omega_o (6) + (0.5)(12)(6)^2 \]

\[ \omega_o = 30.7 \text{ rad/s} \]

In this case the only energy that changes is rotational kinetic energy so

\[ W = \Delta K_{\text{rot}} \]

\[ W = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_o^2 \]
$$v_x = v_{x0} + a_x t$$

$$x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2$$

$$v_x^2 = v_{x0}^2 + 2a_x \Delta x$$

$$K = \frac{1}{2} mv^2$$

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha \Delta \theta$$

$$K = \frac{1}{2} I \omega^2$$
Rolling (without slipping)

Can be thought of as the sum of two motions:

**Rotation** about the center-of-mass (cm)

**Translation** of the cm

\[
K_{\text{rot}} + K_{\text{trans}} = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} m v_{\text{cm}}^2 = K_{\text{rolling}}
\]
A 1 kg object is at rest at the top of an inclined plane and rolls to the bottom of the hill. The top of the hill is 10 m higher vertically than the bottom.

\[ U_{gi} + K_{Transi} + K_{roti} + W = U_{gf} + K_{Transf} + K_{rotf} \]

\[ mgh = \frac{1}{2} mv_{cm}^2 + \frac{1}{2} l_{cm} \omega^2 \]
Some moments of inertia

- **Uniform Sphere**: \(\frac{2}{5}MR^2\)
- **Uniform Ring**: \(MR^2\)
- **Uniform Disk**: \(\frac{1}{2}MR^2\)
- **Uniform Rod**: \(\frac{1}{3}ML^2\)
Which object wins?

Initial energy = final energy
\[ mgh = \frac{1}{2} mv^2_{\text{cm}} + \frac{1}{2} I_{\text{cm}} \omega^2 \]

\[ \omega = \frac{v_{\text{cm}}}{R}, \text{ so this can be rewritten as} \]
\[ mgh = \frac{1}{2} mv^2_{\text{cm}} + \frac{1}{2} I_{\text{cm}} \left( \frac{v_{\text{cm}}}{R} \right)^2 \]

The moment of inertia of symmetric objects (e.g. disk, sphere, hoop) can be written as
\[ I_{\text{cm}} = \beta mR^2 \]
\[ mgh = \frac{1}{2} mv^2_{\text{cm}} + \frac{1}{2} \beta mR^2 \left( \frac{v_{\text{cm}}}{R} \right)^2 \]
\[ mgh = \frac{1}{2} mv^2_{\text{cm}} (1 + \beta) \]
\[ mgh = K_{\text{trans}} (1 + \beta) \]

Written this way, it is straightforward to find what portion of the total energy available goes into rotation, and therefore what portion is translational energy. The object with the biggest portion of its total energy that goes into translation energy will win, since they both have the same amount of energy initially.

The portion of the energy that goes to translation = \( \frac{1}{1 + \beta} \)
The portion of the energy that goes to rotation = \( \frac{\beta}{1 + \beta} \)
A 1 kg disk and a 1 kg hoop are at rest at the top of an inclined plane and roll to the bottom of the hill. The top of the hill is 10 m higher vertically than the bottom. Which one wins?

\[
U_{gi} + K_{Transi} + K_{roti} + W = U_{gf} + K_{Transf} + K_{rotf}
\]

\[
mgh = \frac{1}{2} m v_{cm}^2 + \beta* \frac{1}{2} m v_{cm}^2
\]
A 1 kg disk and a 1 kg hoop are at rest at the top of an inclined plane and roll to the bottom of the hill. The top of the hill is 10 m higher vertically than the bottom. Which one wins?

\[ U_{gi} + K_{Transi} + K_{roti} + W = U_{gf} + K_{Transf} + K_{rotf} \]

\[ mgh = \frac{1}{2} mv_{cm}^2 + \beta \frac{1}{2} mv_{cm}^2 \]

.disk | hoop
A 1 kg disk and a 1 kg hoop are at rest at the top of an inclined plane and roll to the bottom of the hill. The top of the hill is 10 m higher vertically than the bottom. Which one wins?

**BEFORE**

\[ U_{gi} + K_{Transi} + K_{roti} + W = U_{gf} + K_{Transf} + K_{rotf} \]

**AFTER**

\[ mgh = \frac{1}{2} mv_{cm}^2 + \beta* \frac{1}{2} mv_{cm}^2 \]

The hoop has less kinetic energy at the bottom, therefore it is moving slower and it loses.
Torque

Recall that an unbalanced force causes the linear velocity of a particle to change. What causes angular velocity to change?

What happens when an extended object is subject to unbalanced forces?

Torque is VECTOR PRODUCT (which is a maximum when the two forces are perpendicular to each other)

Torque is the product of a Force (vector, \( F \)) that is applied at a point that has a non-zero position (vector, \( r \)) from the axis of rotation.
Torque is a VECTOR PRODUCT

\[ \vec{\tau} = \vec{r} \times \vec{F} \]
\[ \tau = rF \sin \theta \]

RHR:
• Counterclockwise rotation is considered OUT OF THE PAGE  ○
• Clockwise rotation is considered INTO THE PAGE.  ❌
Newton’s 2nd Law

Recall that linear acceleration is caused by unbalanced external forces, and is inversely proportional to the object’s mass:
Newton’s 2\textsuperscript{nd} Law

Recall that linear acceleration is caused by unbalanced external forces, and is inversely proportional to the object’s mass:

$$a = \frac{F_{\text{net}}}{m}$$
Newton’s 2nd Law

Recall that linear acceleration is caused by unbalanced external forces, and is inversely proportional to the object’s mass:

$$a = \frac{F_{\text{net}}}{m}$$

Analogously, angular acceleration is caused by unbalanced external and is inversely proportional to the object’s _______:

$$\alpha = \frac{T_{\text{net}}}{I}$$
Newton’s 2nd Law

Recall that linear acceleration is caused by unbalanced external forces, and is inversely proportional to the object’s mass:

\[ a = \frac{F_{\text{net}}}{m} \]

Analogously, angular acceleration is caused by unbalanced external \textit{torques} and is inversely proportional to the object’s \textit{moment of inertia}:
Newton’s 2\textsuperscript{nd} Law

Recall that linear acceleration is caused by unbalanced external forces, and is inversely proportional to the object’s mass:

\[ a = \frac{F_{\text{net}}}{m} \]

Analogously, angular acceleration is caused by unbalanced external torques and is inversely proportional to the object’s moment of inertia:

\[ \alpha = \frac{\tau_{\text{net}}}{I} \]
\[ x \leftrightarrow \theta \quad v_x \leftrightarrow \omega \quad a_x \leftrightarrow \alpha \quad m \leftrightarrow I \]
\[ t \leftrightarrow t \quad K \leftrightarrow K \quad \vec{F} \leftrightarrow \vec{\tau} \]

**TRANSLATION**
\[
\begin{align*}
v_x &= v_{x0} + a_x t \\
x &= x_0 + v_{x0} t + \frac{1}{2} a_x t^2 \\
v_x^2 &= v_{x0}^2 + 2a_x \Delta x \\
K &= \frac{1}{2}mv^2 \\
\sum \vec{F} &= m\vec{a}
\end{align*}
\]

**ROTATION**
\[
\begin{align*}
\omega &= \omega_0 + \alpha t \\
\theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
\omega^2 &= \omega_0^2 + 2\alpha \Delta \theta \\
K &= \frac{1}{2}I\omega^2 \\
\sum \vec{\tau} &= I\vec{\alpha}
\end{align*}
\]