Constructing BPS quiver and potential

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Based on arXiv:1203.4573, arXiv:1207.6112, arXiv:1211.7071

Motivation

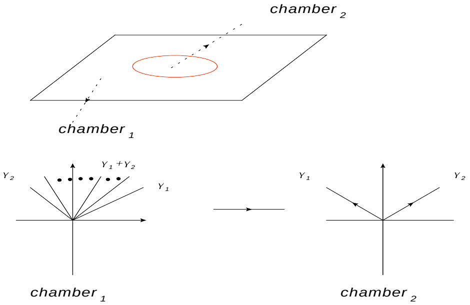
- Find the BPS quiver and potential for theory of class $\mathcal{S}$.
- Get some control about the quiver mutation sequences which are crucial for finding finite BPS chamber.

Generality of BPS particle

- It is labeled by the electric, magnetic and flavor charges (by specifying a IR duality frame)
$$\gamma = (n_e^i, n_m^i, s_f) \tag{1}$$
- It has a central charge given by Seiberg-Witten solution
$$Z(\gamma) = n_e^i a_i + n_m^i a_D^i + s_f m_f \tag{2}$$
- Its mass saturates the BPS bound
$$M = |Z| \tag{3}$$
- BPS particle can decay in crossing the marginal stability wall, which is essential for the consistency of the Seiberg-Witten solution.

Pure $SU(2)$ theory

The BPS spectrum of pure $SU(2)$ theory:



The charge vectors γ_1 and γ_2 has the Dirac product $<\gamma_1, \gamma_2>= 2$, and the BPS spectrum is

- Chamber 1: $\gamma_1 + \gamma_2$ is a vector multiplet. $(\gamma_2, \ldots n\gamma_2 + (n-1)\gamma_1 \ldots, \gamma_1 + \gamma_2, \ldots (n-1)\gamma_2 + n\gamma_1 \ldots, \gamma_1)$, these are hypermultiplet
- Chamber 2: (γ_1, γ_2), all hypermultiplets

Tools for finding the BPS spectrum

The Seiberg-Witten solution is not that useful to find the BPS spectrum of a general $\mathcal{N}=2$ field theory. One need some new structures:

- Geometric approach: Spectral networks, etc (Gaiotto, Moore, Neitzke)
- Quiver and potential (Denef, Cecotti, Vafa, et al)

These two approaches are closed related (one can get a quiver from the spectral network of higher rank theory, and quiver is more conveniently to organize things), and the most powerful way of counting the spectrum is to combine the geometric and the quiver approach.

BPS quiver

The BPS quiver could be interpreted from the known spectrum (Cecotti, Vafa, etc):

- Choose a half plane of central charge, and take a canonical basis γ_i such that the charge vectors for all the other stable particles are expanded with **Positive** integer coefficient

$$\gamma = \sum n_i \gamma_i, \quad n_i \geq 0 \tag{4}$$

- The BPS quiver is formed by taking the Dirac product of the basis

$$\epsilon_{ij} = < \gamma_i, \gamma_j > \tag{5}$$

γ_1 and γ_2 is the canonical basis for pure $SU(2)$ theory, and the BPS quiver is

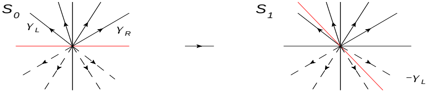
$$\bullet \rightrightarrows \bullet \tag{6}$$

Quiver mutations

- The number of quiver nodes is $R = 2n_r + n_f$, and the rank of the quiver is $2n_r$.
- The quiver is not unique. If another half plane is chosen, the canonical basis would be different, and the new basis is

$$\begin{aligned} \gamma_k^{'} &= -\gamma_k \\ \gamma_i^{'} &= \gamma_i + [\epsilon_{ik}]_+ \epsilon_k \end{aligned} \tag{7}$$

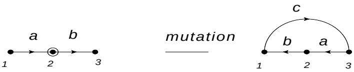
The new quiver is formed using the new basis, and this operation on the quiver is called quiver mutation!



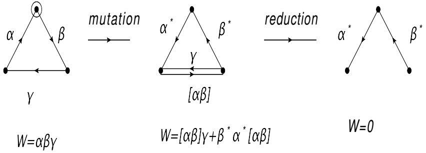
The quiver mutation has a very nice graphic representation,

- : Reverse the quiver arrows attahced on the quiver node k.
- : If there is an oriented path $i \to k \to j$, then add a new arrow $i \to j$.
- : Remove the two cycles of the quiver arrows.

This is exactly the Seiberg duality. And BPS quiver is a class of quivers related by the quiver mutations.



Since the two quivers related by the mutation has quite different representation theory (or quite different moduli space). As in the context of Seiberg duality, one need to add potential to the quiver to say they are equivalent , as discovered by (Derksen, Weyman, Zelevinsky).
The transformation rule of the potential under mutation is actually the same as those used in Seiberg duality:



- So what we need is a family of quiver and potential (Q, W) related by the mutations.
- The question is: How to find such family of pairs for any given four dimensional $\mathcal{N} = 2$ theory! This seems to be a hard problem since we do not know anything about the BPS spectrum of most theories!
- I will show that one can find the required pairs for the theory of class $\mathcal{S}$ using some very simple combinatorial rules! (One can also use the geometric engineering method, see Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Jan Manschot, Gregory W. Moore, Yan Soibelman)

Theories of class $\mathcal{S}$

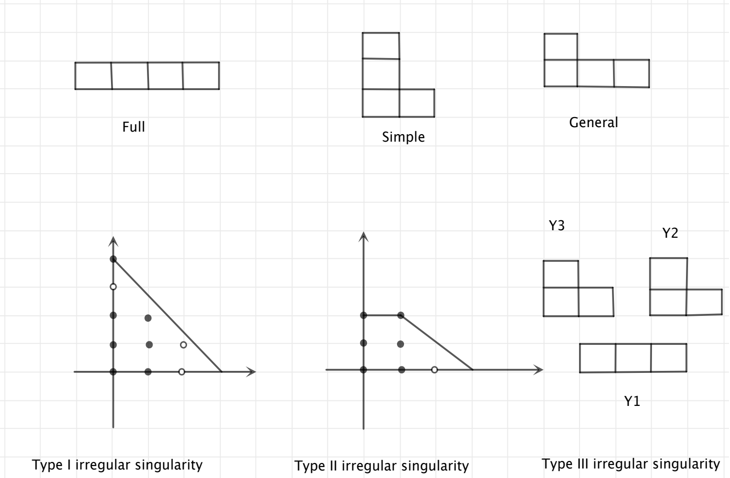
A huge class of N $= 2$ theory can be engineered from six dimensional A_{N-1} (2, 0) theory compactified on a Riemann surface with regular singularities and irregular singularities. The data is the following:

- M_{g,p_i,b_j} , where g is the genus, p_i is the regular singularity, and b_j is the irregular singularity.
- p_i is classified by the Young Tableaux (Gaiotto, 0904.2715, Gaiotto, Moore, Neitzke, 0907.3987), b_j is classified by a Newton Polygon or a sequence of Young Tableax (DX, 1204.2270).

Many properties of these theories including the Seiberg-Witten curve, 3d mirrors, central charges, extended objects can be answered even the theory does not have a Lagrangian.

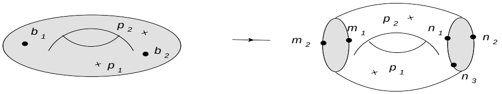
Puncture gallery

Here are some examples of the punctures.



BPS geometry

We could use the above geometric construction to find the BPS quiver. It is essential to replace the irregular singularity with a boundary with marked points which are also classified by Young Tableaux (DX, 1207.6112): i.e. the type I irregular singularity is replaced by a sequence of full and simple punctures on the boundary. This is done by doing the stokes analysis about the irregular singularity. In the A1 case (GMN 0907.3987), there is only one type of Young Tableaux, so all the marked points are the same.

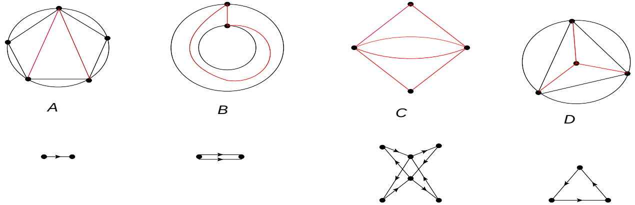


Review on A1 case

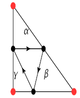
The construction starts with the triangulation of the BPS geometry.

- Assign a quiver node to each internal edge of the triangulation.
- There is an arrow between two quiver nodes if they are in the same triangle, and one remove all the two cycles.

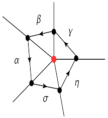
The quiver in this class is mutation finite, so it is rather simple from the quiver mutation point of view but the BPS counting can be quite difficult.



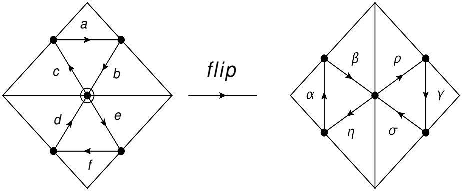
The potential can also be easily defined, and one can change the triangulations using the local flips, and the quiver and potential transform in the desired way



$W=\alpha\beta\gamma$



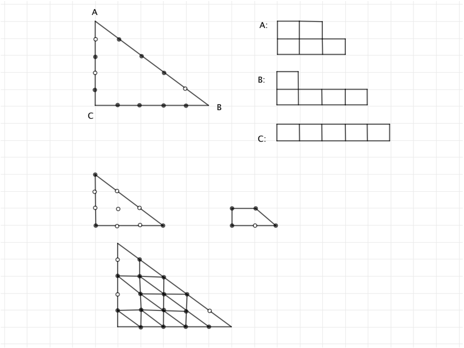
$W=\alpha\beta\gamma\eta\sigma$



$W=abc+def$ $\quad$ $W=\alpha\beta\eta+\rho\gamma\sigma$

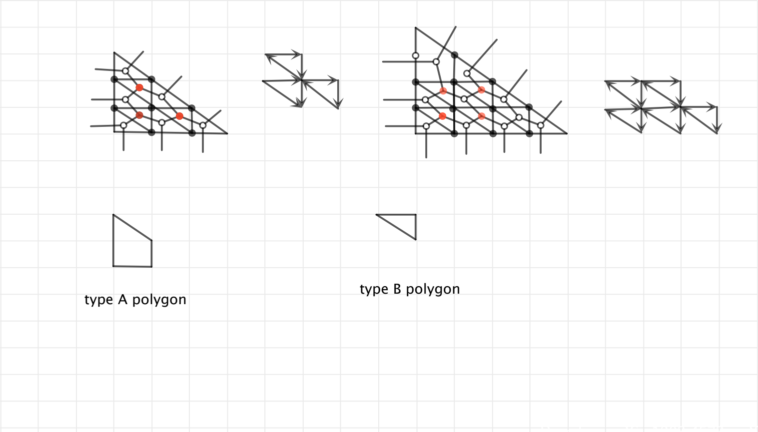
Step 1: tessellation of triangle

For the higher rank theory, we start with a regular triangulation and a choice of the cyclic path connecting all the punctures. One one find a tessellation about a triangle with three generic punctures (use the rules by Benini, Bevenuti and Tachikawa):



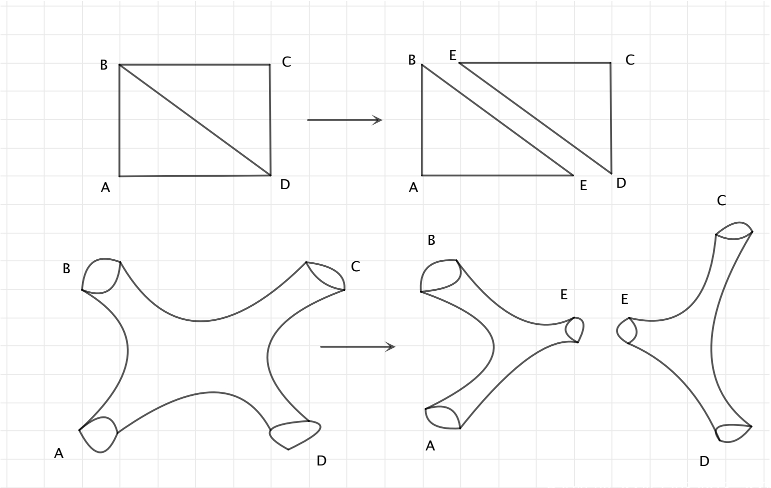
Step 2: Construct a bipartite network and quiver

- Put a black (white) vertex on a type A (B) polygon, and get a bipartite network
- Assign a quiver node to each face and quiver arrows from the black vertex, assign a potential to each vertex



Step 3: Find the decoration on the internal edge and gluing

Given a choice of cyclic path, we know how to decorate the dots on the boundary, what about the internal edge? Let's use the S duality, and luckily one can read the puncture from the data of the Young Tableaux.



The idea is the following: let's define a vector $p^i = i - s_i$ from the puncture, where i is the height of the ith box, then the new puncture in the degeneration limit is found from the formula

$$p_E^i = \min(i-1, p_A^i + p_B^i, p_C^i + p_D^i) \tag{8}$$

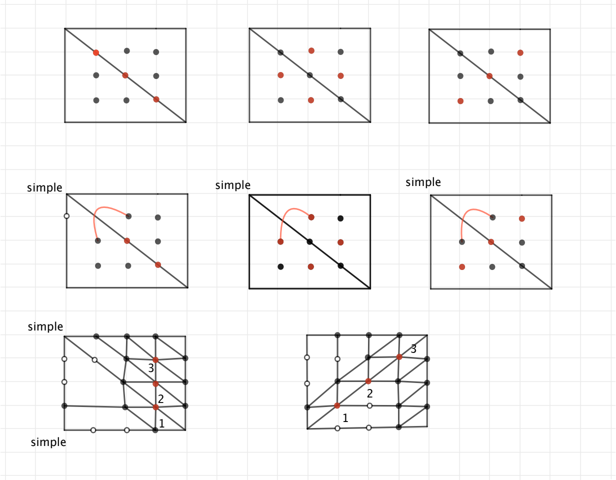
For example, if A and B are simple punctures, then $p_A = p_B = (0, 1, \ldots, 1)$, and then $p_E = (0, 1, 2, 2, 2, \ldots)$, and the Young Tableaux is $(N-2, 1, 1)$.

Once the internal decoration is found, one can construct a dot diagram and then quiver and potential (from bipartite network)!

- The total number of quiver nodes are $2n_r + n_f$.
- The quiver is generically mutation infinite, but they are related by the flips, which is quite remarkable.

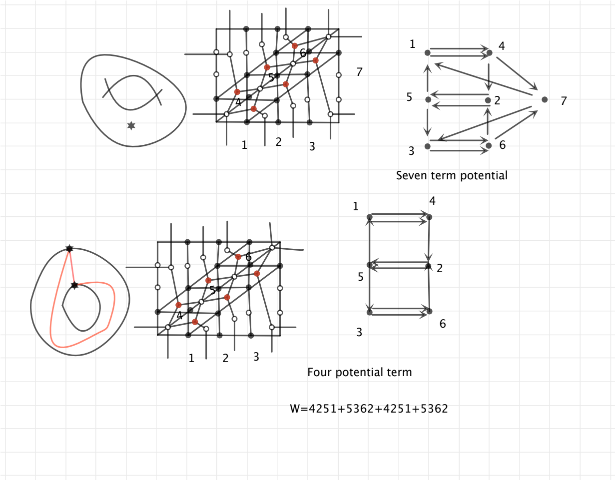
Flip

The quivers from different triangulations of the quadrilateral are related by a sequence of quiver mutations (provided that one do not have the bad configuration at the corner)



Examples

Let's construct the BPS quiver and potential for pure $SU(N)$ theory and $SU(N)$ with one adjoint.



Conclusion

We have constructed the BPS quiver and potential for the theory of class $\mathcal{S}$, and one can use the following tools to find the BPS spectrum

- Quiver representation theory and θ stability
- Cluster algebra and cluster category
- Maximal green mutations (powerful in finding the finite chamber), and the flips are very useful
- ....