Spontaneous and induced decay of false vacuum

Mikhail Voloshin
FTPI, University of Minnesota
• Introduction. False vacuum.

\[ V \phi \phi \epsilon \]

\[ d \text{ space-time dimensions:} \]
Gain in volume energy: \(-\text{Volume} \times \epsilon \propto \epsilon R^{d-1}\)
Loss in surface energy: \(\text{Area} \times \mu \propto \mu R^{d-2}\)

\[ R_c = (d - 1) \frac{\mu}{\epsilon} \]
The rate of critical bubble formation (per unit time \( \times \) volume) \( w_0 \sim \exp(-\text{Action}) \).

- Euclidean-space calculation

Decay rate = imaginary part \((\times(-2))\) of the false vacuum energy. The path integral

\[
Z = \mathcal{N} \int e^{-S[\phi,...]} \, D\phi \ldots = \exp(-E_{\text{vac}}T)
\]

\( \Rightarrow w_0 = 2 \text{Im}(\ln Z)/VT \).

- Bounce

Stationary configuration: \( O(d) \)-symmetrical solution of field equations with \( \phi \rightarrow \phi_+ \) at \( r \rightarrow \infty \) and \( \phi \approx \phi_- \) at \( r = 0 \).

\[
w_0 \, d^d \, z = \left| \frac{\text{det}'(S^{(2)})}{\text{det}(S_0^{(2)})} \right|^{-1/2} \exp(-S_B) \left( \frac{S_B}{2\pi} \right)^{d/2} d^d \, z
\]
Pre-exponent

\[ w_0 = \Gamma \exp(-S_{cl}) \]

- \( d = 2 \)
  \( \Gamma = \frac{\epsilon}{2\pi} \) in a model with no fermions
  \( \Gamma = 2^N \frac{\epsilon}{2\pi} \) for a model with \( N \) fermions having zero mode on the kink (in the \( \epsilon \to 0 \) limit). \( 2^N \) = the number of final states for kink-antikink.

Some details on \( d = 2 \):
- \( \gamma \) — closed curve in \( d = 2 \).
- Effective action: \( S[\gamma] = \mu P[\gamma] - \epsilon A[\gamma] \).

\[ Z_1 = \int \exp(-S[\gamma]) \mathcal{D}\gamma \]

\( \mu \) and \( \epsilon \) are the renormalized parameters supplied by the ‘microscopic’ theory.

Stationary curve: circle with \( R = \mu/\epsilon, S_{cl} = \pi \mu^2/\epsilon \).

Let \( r(\theta) \) be the polar parametrization of \( \gamma \). Hamiltonian form:

\[ Z_1 = \int \exp \left( -\int p \, dq + \int H \, d\theta \right) \frac{\mathcal{D}p \, \mathcal{D}r}{2\pi} \]

\[ p = \mu \dot{r}/\sqrt{r^2 + \dot{r}^2}, \quad H = \frac{1}{2} \epsilon r^2 - r \sqrt{\mu^2 - p^2} \]

\( |p| < \mu \) — no self-intersections of \( \gamma \).

Canonical transform \((r, p) \to (q, p)\) with \( q = r - \sqrt{\mu^2 - p^2}/\epsilon \):

\[ H(q, p) = \frac{p^2}{2\epsilon} + \frac{\epsilon}{2} q^2 - \frac{\mu^2}{2\epsilon} \]
which up to the constant $-\mu^2/(2\epsilon)$ is a Euclidean-space Hamiltonian for a harmonic oscillator with the frequency $\omega^2 = -1$.

$\Rightarrow$ The path integral for $Z_1$ is Gaussian in a finite neighborhood of the stationary point $p = 0, q = 0$.
$\Rightarrow w_0 = \frac{\epsilon}{2\pi} \exp(-\pi \mu^2/\epsilon)$ is exact up to higher exponents. (No power corrections in $\epsilon/\mu^2$)

One known exact case (M. Stone ’76): Sine-Gordon staircase

$$\frac{1}{2} (\partial_\mu \phi)^2 + \frac{\alpha}{\beta^2} \cos \beta \phi + J \phi \leftrightarrow i \bar{\psi} \gamma \cdot \partial \psi - \frac{1}{2} g j^\mu j_\mu + m \bar{\psi} + e A^0 j^0$$

$\beta^2/4\pi = (1 + g/\pi)^{-1}, \partial_1 A^0 = J, e = 2\pi/\beta$.

Schwinger process: pair creation in external field $E$. At $\beta^2 = 4\pi$ the Thirring model is free, $g = 0, \Rightarrow$ exact result

$$w_0 = -\frac{\epsilon}{2\pi} \ln \left[ 1 - \exp \left( -\pi \frac{\mu^2}{\epsilon} \right) \right]$$
• $d = 3$ The ‘low-energy’ effective action for 2-d closed surface $\gamma$ in 3-d

\[ S = \mu \text{Area}[\gamma] - \epsilon \text{Volume}[\gamma] \]

is not renormalizable. Still some universality remains:

\[ w_0 = \frac{A}{\epsilon^{7/3}} \exp(-S_{cl}) \]

$A$ depends on the parameters (masses, couplings) of the ‘microscopic’ theory, but not on $\epsilon$.

Specific $\phi^4$ model: G. Münster and S. Rotsch ‘00, general case MV ‘04

• Catalysis by presence of a particle. (Particle decay → true vacuum.)
Energy transfer when the particle field has zero mode on the wall. (Boson of the master field, or fermion.)
Initial state $\delta E = m$ (the particle mass). Final state: $\delta E = 0$ (particle ‘rides’ as a bound state on the bubble wall).

Decay rate: $\Gamma = K w_0$. $K$ - catalysis factor.

$d = 2$
Capillarity problem. $2\mu \cos \alpha = m$

$$S_{eff} = \frac{\mu^2}{\epsilon} \left[ 2 \arccos \left( \frac{m}{2\mu} \right) - \frac{m}{\mu} \sqrt{1 - \frac{m^2}{4\mu^2}} \right]$$

Tunneling through the barrier $2\mu - 2\epsilon R$ at energy $E = m$.

Same applies to collisions at energy $E$. 
$|A_+| \sim \exp \left[ -\frac{\bar{\mu}^4}{\epsilon^3} (b(E) + c(E)) \right]$
Figure 1: The barrier penetration function $b(E)$, the excitation function $c(E)$, and their sum vs. $w = E \tilde{\epsilon}^2 / \tilde{\mu}^3$. At the point $w_c = 4/27$ and beyond the barrier disappears, hence $b(E) = 0$ and the sum coincides with $c(E)$.

For $mR \ll S_B$, but still $mR \gg 1$: $K = C \cdot \exp(2mR)$.

Calculation of $C$ is the subject.
- $mR \ll S_B$ - arbitrary $d$.
- $m \approx 2\mu$ in $d = 2$. 
Boson

Consider the particle propagator in the \( \phi_+ \) vacuum (\( \sigma(x) = \phi(x) - \phi_+ \)):

\[
D(x, y) = \frac{1}{Z} \int \sigma(x) \sigma(y) e^{-S[\phi,...]} \mathcal{D}\phi \ldots
\]

Bounce contribution at \( L = |x - y| \gg R \)

\[
\delta D(x, y) = \frac{i}{2} w_0 \int d^d z \, F(x - z, y - z) \, D_0(x - z) \, D_0(y - z) ,
\]

\( D_0(x) \) - free propagator. Use saddle point in the \( dz \perp \) integration. Then at \( L \gg R \)

\[
\delta D(x, y) = \frac{i}{2} w_0 \, F_0 \int d^d z \, D_0(x - z) \, D_0(y - z) ,
\]

with \( F_0 \) given by the alignment in Figure b). Compare with effect of \( m^2 \to m^2 + \delta m^2 \):

\[
\delta_m D(x, y) = -\delta m^2 \int d^d z \, D_0(x - z) \, D_0(y - z)
\]
\[ \delta m^2 = -(i/2) w_0 F_0 \], which corresponds to the particle decay rate \( \Gamma = F_0 w_0/(2m) \) ⇒ the catalysis factor \( K \) is found as

\[ K = \frac{F_0}{2m} \]

For the bosons of the master field \( F_0 \) is found from asymptotic form of (classical) bounce field profile \( \phi(r) - \phi_+ \to -2v \exp[-m(r - R)] \to CD_0(r) \).

\[
D_0(r) = \frac{m^{d/2-1}}{(2\pi)^{d/2} r^{d/2-1}} K_{d/2-1}(mr)
\]

\[ \Rightarrow C = -4(2\pi)^{d/2-1} m^{(3-d)/2} R^{(d-1)/2} v e^m R \]

\[ F_0 = C^2 \Rightarrow \]

\[ K = 2^{d+1} \pi^{(d-3)/2} \Gamma \left( \frac{d+1}{2} \right) m^{2-d} v^2 V_{d-1} e^{2m R} \]

\( V_{d-1} = \pi^{(d-1)/2} R^{d-1}/\Gamma[(d + 1)/2] \) is the spatial \((d - 1\) dimensional) volume of the critical bubble.

Enhancement of \( K \): classical \( \exp(2mR) \) and an extra factor \( m^{2-d} v^2 \) - inverse of the (small) coupling constant for \( \phi \).
• Infrared complication in $d = 2$
Soft modes: distortion of the shape of the bounce. Eigenvalues $\lambda_n = c_n/R^2$.

$d = 2$ normalized eigenmode ($\mu \sim mv^2$):

$$\sigma_n(r) \sim \frac{m v}{\sqrt{\mu R}} e^{-m(r-R)} \sim \sqrt{\frac{m}{R}} e^{-m(r-R)}$$

⇒ mode contribution to $\delta D(r_1, r_2)$:

$$\frac{R^2}{c_n} \sigma_n(r_1)\sigma_n(r_2) \sim c_n^{-1} (m R) e^{2mR} e^{-m(r_1+r_2)}$$

Classical part: $\sim v^2 e^{2mR} e^{-m(r_1+r_2)} \Rightarrow$

$$\frac{\text{Mode contrib.}}{\text{Classical}} \sim \frac{m R}{v^2}$$

Although, as expected, suppressed by the small coupling $1/v^2$, is infrared unstable at large $R$. 
Solution

Effective action for the soft modes (polar-coordinate parametrization of the bounce shape, \( (r(\theta), \theta) \))

\[
S = \int_0^{2\pi} \left( \mu \sqrt{r^2 + \dot{r}^2} - \frac{1}{2} \epsilon r^2 \right) d\theta = \frac{\pi \mu^2}{\epsilon} + \int_0^{2\pi} \frac{\epsilon}{2} (\rho^2 - \rho^2) d\theta + O(\rho^4)
\]

\( R = \mu/\epsilon, \rho(\theta) = r(\theta) - R, \dot{\rho} = d\rho/d\theta. \)

Modes: \( \lambda_n \propto (n^2 - 1). \)

One negative mode: \( \rho_0 = 1/\sqrt{2\pi}. \)

Two types of \( n > 0 \) modes:

\[
\rho_n^{(1)} = \frac{1}{\sqrt{\pi}} \cos n\theta, \quad \text{and} \quad \rho_n^{(2)} = \frac{1}{\sqrt{\pi}} \sin n\theta; \quad (n = 1, 2, \ldots)
\]

Only the fluctuations of the vertical size of the bounce contribute to \( \delta D(x, y): \)

\[
\langle [\rho(0) + \rho(\pi)]^2 \rangle \propto \sum_n \frac{[\rho_n(0) + \rho_n(\pi)]^2}{n^2 - 1}
\]

The sum \( \rho(0) + \rho(\pi) \) is not vanishing only for the negative mode and for the positive modes of the first type, \( \rho_n^{(1)} \), with even \( n \), i.e. \( n = 2k \). Thus

\[
\langle [\rho(0) + \rho(\pi)]^2 \rangle \propto -\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = 0
\]

The theory cures itself from the infrared problem by cancellation between one negative and the sum over the positive modes.
• Fermionic case. $d = 2$.

If a fermion field is present in the theory (no actual fermions in the false vacuum), such that the fermion mass $m(\phi)$ changes sign between $\phi_+$ and $\phi_-$, the fermion has a zero mode on the bubble wall and affects the spontaneous decay rate of the false vacuum: in $d = 2$ it makes $w_0 \rightarrow 2w_0$. Two (degenerate) final states: $F(\text{kink}, \text{antikink}) = (+1/2, -1/2)$ and $F(\text{kink}, \text{antikink}) = (-1/2, +1/2)$.

• Fermion present in the false vacuum.

Consider the bounce contribution to the fermion Green function $G(x, y) = \langle \psi(x) \bar{\psi}(y) \rangle$.

Fermion zero mode, $[\sigma_i \partial_i + m(\phi)] \psi_0 = 0$

$$\psi_0(r, \theta) = C_f \sqrt{\frac{R}{r}} \exp \left\{ - \int_R^r m[\phi(r')] dr' \right\} \chi(\ell) \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}$$

$\chi(\ell)$ is a one-dimensional fermion field living on the bounce boundary and (nominally) depending on the length parameter $\ell = R\theta$ along the boundary. Classical equation for $\chi$: $\dot{\chi} = 0$. $C_f$ is the normalization factor:

$$2 C_f^2 \int \exp \left[ -2 \int_R^r m(\phi) dr' \right] dr = 1.$$ 

Switch to $\tilde{C}_f$:

$$C_f \exp \left\{ - \int_R^r m[\phi(r')] dr' \right\} \rightarrow \tilde{C}_f \exp [m_f(R - r)]$$

Generally

$$\tilde{C}_f^2 = \frac{m_f}{2} f \left( \frac{m_f}{m} \right)$$

$f$ - dimensionless function of $m_f/m$ with $m$ standing for other mass parameters in the false vacuum.
For \( m_f/m \to 0, \ f(m_f/m) \to 1. \) In a \( \phi^4 \) theory with \( m = \) the boson mass

\[
f(u) = \frac{2^{2u}}{\sqrt{\pi}} \frac{\Gamma(u + 1/2)}{\Gamma(u + 1)}
\]

Contribution of the zero mode to \( G(x, y) \):

\[
\delta G(x, y) = -\frac{i}{2} \frac{w_0}{2} d^2 z \tilde{C}_f^2 e^{2m_f R} R \frac{e^{-|x-y|}}{\sqrt{|x-z||y-z|}} (1 + \sigma_1) g(0, \pi R)
\]

with \( g(\ell_1, \ell_2) = \langle \chi(\ell_1)\chi^\dagger(\ell_2) \rangle \) the propagator of \( \chi \).

\( g(\ell_1, \ell_2) = (1/2) \text{sign}(\ell_1 - \ell_2) \Rightarrow g(0, \pi R) = -1/2 \)

Compare with \( m_f \to m_f + \delta m_f \):

\[
\delta_m G(x, y) = -\delta m_f d^2 z G_0(x - z) G_0(z - y) \to -\delta m_f d^2 z \frac{m}{4\pi} (1 + \sigma_1) \frac{e^{-|x-y|}}{\sqrt{|x-z||y-z|}}
\]

\( G_0(x, y) = \frac{1}{2\pi} (-\sigma_i \partial_i + m) K_0(m_f |x - y|) \Rightarrow \)

\[
\Gamma_f = \frac{\pi}{2} f \left( \frac{m_f}{m} \right) R w_0 \exp(2m_f R) = \frac{\mu}{2} f \left( \frac{m_f}{m} \right) \exp \left( -\frac{\pi \mu^2}{\epsilon} + 2m_f R \right)
\]

\( w_0 = \epsilon/\pi \) \( \exp(-\pi \mu^2/\epsilon) \) and \( R = \mu/\epsilon \)

For the fermion indeed \( K_f \sim R \exp(2m_f R) \).
Meson decay in sine-Gordon model

**Weak coupling**

\[
L_{SG} = \frac{1}{2} (\partial \phi)^2 + \frac{\alpha}{\beta^2} \cos(\beta \phi) + \left(\frac{\epsilon \beta}{2\pi}\right) \phi
\]

The catalysis factor for a boson is

\[
K = \frac{32}{\beta^2} \frac{\mu}{\epsilon} e^{2m_b \mu / \epsilon}
\]

\(m_b\) = boson mass.

**Strong coupling**

Equivalent: Thirring model in external electric field

\[
L_{Th} = i \bar{\psi} \partial_\nu \gamma^\nu \psi - \frac{1}{2} g j^\nu j_\nu + \mu \bar{\psi} \psi + A_0 j_0
\]

\[
\frac{\beta^2}{4\pi} = (1 + \frac{g}{\pi})^{-1}, \ j_\nu = \bar{\psi} \gamma_\nu \psi, \ \mu = \text{soliton mass in the sine-Gordon model}, \ \partial_x A_0 = \epsilon.
\]

Small \(g\): the SG boson is a shallow bound state of fermion-antifermion, \(m_b = 2\mu - \mu g^2\).

The near-threshold dynamics of the fermion-antifermion (soliton-antisoliton) pair can be described by the nonrelativistic Hamiltonian

\[
H = \frac{\vec{p}^2}{\mu} - \epsilon x - 2g \delta(x)
\]

Boson-induced vacuum decay = ionization of the bound state in \(U(x) = -2g\delta(x)\) by ext. electric field \(\epsilon\).
Equation for the Green function \((E = -\kappa^2/\mu)\):

\[
G(0, 0; -\kappa^2/\mu) = \frac{G_\epsilon(0, 0; -\kappa^2/\mu)}{1 - 2g G_\epsilon(0, 0; -\kappa^2/\mu)}
\]

with \(G_\epsilon(x, y; E)\) the Green function in linear potential \((-\epsilon x)\)

\[
G_\epsilon \left(0, 0; -\frac{\kappa^2}{\mu}\right) = \int_0^\infty \sqrt{\frac{\mu}{4 \pi \tau}} \exp \left(\frac{\epsilon^2}{12 \mu} \tau^3 - \frac{\kappa^2}{\mu} \tau\right) d\tau
\]

The pole (in \(\kappa\)) is determined by

\[
2g G_\epsilon \left(0, 0; -\frac{\kappa^2}{\mu}\right) = 1
\]

\[
\tau_0 = 2\kappa/\epsilon. \text{ The decay rate (due to ionization):}
\]

\[
\Gamma = 2 \mu g^2 \exp \left(-\frac{4}{3} g^3 \frac{\mu^2}{\epsilon}\right)
\]