A warm-up for solving noncompact sigma models: 
The Sinh-Gordon model

Jörg Teschner

Based on

- A. Bytsko, J.T., hep-th/0602093,
- J.T., hep-th/0702122
Exponential Interaction

\[ \cdots + \left( \cdots \right) \partial_{\bar{z}} \vartheta + \left( \cdots \right) \partial_{z} \vartheta + \cdots = H \]

Writing \( z = e^{\theta} \) world-sheet Hamiltonian of the form

\[ \partial_{x} p + \partial_{\bar{z}} p \frac{\bar{z}}{z} = \zeta p \]

On the other hand: Hope for Integrability! (Bena, Polchinski, Roiban)

Building blocks: nonlin. sigma-models on super-groups.

Goal: Understand string theory on AdS-spaces.
Proposal has passed highly nontrivial tests!

Proposal for 2 \rightarrow 2 scattering matrix (for all $y$):

Elementary excitations ("magnons") $8B \oplus 8F$.

\begin{align*}
\infty & \leftrightarrow f \leftrightarrow \infty
\end{align*}

Proposal for factorized scattering theory in infinite volume $\mathcal{H}^\infty$ is integrable.

Conjecture: \( H^{G,F} \) is integrable.

\[
\left( \cdots + (\cdots)_{\phi_{-2 \zeta}} \theta + (\cdots)_{\phi_{2 \zeta}} \theta + \cdots \right) \phi \int_0^{2\pi R} = \mathcal{H}^{G,F}
\]

Gauge-fixing: \( h^y = h^x = f - \mathcal{H} \), \( \infty \leftrightarrow f \)

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Important problem: Spektrum for finite $R$, finite $J$?

- Pretend magnons are free except for magnons crossing ($\phi$-like interactions),

  \[ e^{ip_a R} = \prod_{p \neq q} \mathcal{S}(p_a, p_b) Y_{a_b} \]

- Ansatz for coord. space wave-fct.: Plane-waves + crossings ($\phi$-matrix),

- Asymptotic Bethe ansatz (Arutyunov, Frolov, Staudacher)

- Expect corrections to As. Bethe ansatz

  Problem: Interactions are not $\phi$-like (vacuum polarization)
Classes of integrable models:

(A) Compact integrable models.
- Models associated to compact groups: WZNW, \( \sigma \)-models, XXX, XXZ, Ising, Interactions \( e \)•

(B) Noncompact integrable models - Real type
- Models associated to non-compact groups: WZNW, \( \sigma \)-models, XXX, XXZ, Liouville theory, Sine-Gordon, \( \phi \) theory, Toda...

But: Class (B) is very different from (A):

• Condensed matter: Integer quantum Hall, electron systems with disorder,

• Gauge theory - via AdS-CFT correspondence,

• String theory: Strings on noncompact target spaces (black holes, cosmology),

- Prediction: Bethe ansatz fails in class (B)!

Class (B) is important:
Consider prototype: Sinh-Gordon model (on circle with circumference $R$)

First example for QFT from class $(B)$ where spectrum was understood for finite volume $\mathcal{H}$:

- **Spectrum:** Conserved quantities, classification of eigenvectors - today!

  \[ \frac{0}{0} \sin \theta \frac{1}{\sin \theta - \frac{0}{0}} = (\theta) \mathcal{S} \]

- **Fields, correlation functions:** ...

- **Scattering:** $S$-matrix factorizes into two-particle scattering

  \[ S^{\infty} = \frac{\sinh \theta}{\sin \theta} \]

- **Fields, correlation functions:** ...

Finite volume $\mathcal{H}$:

- **Spectrum:** One massive particle, $E = m \cosh \theta$, $p = m \sinh \theta$.

Scattering: $S$-matrix factorizes into two-particle scattering

\[ S^{\infty} = \frac{\sinh \theta}{\sin \theta} \]

Finite volume $\mathcal{H}$:

- **Spectrum:** Conservation of quantities, classification of eigenvectors - today!

\[ \{ (\phi \phi) \frac{\cosh}{\cosh} + \zeta (\phi \phi) \frac{\cosh}{\cosh} \frac{1}{1} + \zeta (\phi \phi) \} \]
First step: Construct integrable lattice regularization of the Sinh-Gordon model:

1. Definition of $\mathcal{H}$:

   Discretize Sinh-Gordon variables as
   
   \[ u \rightarrow \Phi \quad \text{and} \quad \Phi \rightarrow \text{discrete versions} \]

   Quantize:
   
   \[ \nabla u = x \quad \text{and} \quad (x) \Phi \rightarrow u \Phi \quad \text{and} \quad \nabla (x) \Pi \rightarrow u \Pi \]

   Set $\mathcal{G} = \{ T_0, T_1, \ldots \}$ for commuting conserved charges.

   Hamiltonian $H$.

   Hilbert space $\mathcal{H}$.

   Hilbert space $\mathcal{H}$ of (integrable) regularized model:

   
   \[
   \mathcal{H} = \bigotimes_{n=0}^{N} L^2(\mathbb{R})^2
   \]

   Set $\mathcal{G}$ of commuting conserved charges.

   Hamiltonian $H$.

   Hilbert space $\mathcal{H}$.
The operators $\sum_{u} u w(n q u \varpi - \vartheta -)0^w \geq \sum_{u} n N q u \varpi = (n) \forall \mathcal{I} = (n) \mathcal{I}$

Consider the one-parameter family of operators:

$\frac{v g \wedge q = \vartheta}{(s-\vartheta)} \left(\frac{(s-\vartheta)}{(s+\vartheta)} + \vartheta + \mathcal{I}\right) = (n) \mathcal{I}$

$\frac{(u \Phi \sigma - n q u \varpi) \varphi \sin \sigma = (n) \mathcal{I}}{(u \Phi \sigma + n q u \varpi) \varphi \sin \sigma = (n) \mathcal{I}}$

where

$(n) \mathcal{I} \cdot (n) \mathcal{I} \cdot \cdots \cdot (n) \mathcal{I} = \left(\begin{array}{cc}(n) \mathcal{I} & (n) \mathcal{I} \\ (n) \mathcal{I} & (n) \mathcal{I} \end{array}\right) = (n) \mathcal{I}$

2. Construction of $\mathcal{I}$
There exists an operator $H$ which has the following properties.

There exists an operator $H$ which has the following properties.

3. Construction of $H$

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3. Construction of $H$
G-operators and separation of variables:

- non-compactness of target space
- exponential interactions $\phi \in \mathcal{H}$

Reasons:
- no normalizable solution $u \in \mathcal{H}$

Bad news: The Bethe ansatz fails:

$0 = \mathcal{B}(n) u \phi \in \mathcal{H}$

Since interactions involve real exponentials $e^{b'}$:

$\mathcal{B}(n) u \phi \in \mathcal{H}$

Reasons:
- exponential interactions
- non-compactness of target space!

Strategy: Diagonalization of $\mathcal{H}$

Good news: We do have integrable lattice regularization

$\phi \in \mathcal{H}$
Assume we have an operator $\mathcal{Q}$ related to $\mathcal{T}$ via the Baxter equation

\[
\mathcal{Q}(u) \mathcal{T}(u) = \mathcal{T}(u) \mathcal{Q}(u)
\]

which further satisfies

\[
(q_i + n) \mathcal{Q}_N ((n)p) + (q_i - n) \mathcal{Q}_N ((n)p) = (n) \mathcal{T}(n) \mathcal{Q}
\]

Eigenvalues of $\mathcal{Q}$ must satisfy the Baxter equation.

\[
\begin{bmatrix}
\mathcal{Q}((n)\mathcal{Q}(a)) = (n)\mathcal{Q}(a) \\
\mathcal{Q}((n)\mathcal{Q}(a)) = (a)\mathcal{Q}(n)\mathcal{Q}
\end{bmatrix}
\]

Diagonalization of $\mathcal{Q}$ is repackaging of conserved quantities.

\[
(n)\mathcal{Q}((n)\mathcal{Q}(v)) = (n)\mathcal{Q}((n)\mathcal{Q}(u))
\]

Explicit construction of $(n)\mathcal{Q}$ by Bytsko, J. T.

Properties of $\mathcal{Q}$:

- $\mathcal{Q}$ is normal.
- $\mathcal{Q}$ is repackaging of conserved quantities.
- $\mathcal{Q}$ satisfies the Baxter equation.

Quantization conditions:

The analytic and asymptotic properties of $\mathcal{Q}$:

1. $(n)\mathcal{Q}(a) \perp (n)\mathcal{Q}(c)
2. $(n)\mathcal{Q}(a) \mathcal{Q}(n)\mathcal{Q}(q)
3. $(n)\mathcal{Q}(a) \mathcal{Q}(n)\mathcal{Q}$ is normal.
A function $t(u)$ can be an eigenvalue of the transfer-matrix $T(u)$ only if there exists a function $q_t(u)$ which satisfies

\[
\begin{align*}
\left. \begin{array}{l}
\frac{\zeta}{\pi} < |(n)\Re|, & \infty \leftarrow |n| & \text{for } nN\zeta - n \in \mathbb{N}^{+} \land \Re - \varnothing \\
\frac{\zeta}{\pi} > |(n)\Re|, & \infty \leftarrow |n| & \text{for } nN\zeta - n \in \mathbb{N}^{-} \land \Re + \varnothing \\
\end{array} \right\} \sim (n)^{\Re} (\Re) \\
\left. \begin{array}{l}
\forall s \in \mathbb{C} \ni \text{meromorphic in } \mathbb{C}, \text{ with poles in } s \\
q_t(u) \in (n)^{\Re} (\Re) \\
\end{array} \right\} \text{ where } \frac{d}{du}((n)p) = (n)p \\
(\ Re + \iota n)(q + n)\Re - \varnothing \left( \frac{\psi}{\sqrt{\mu}} \right) + 1 = (n-p) = (n-p) \\
\end{align*}
\]

\[
\begin{align*}
\Re + (n)\Re & = (n)\Re + (n)\Re \\
\end{align*}
\]
Separation of Variables

Main idea (Sklyanin):
Diagonalize \( B(u) \), parametrize eigenvalues as

\[
Y \sinh 2b(u) = \prod_{k=1}^{N} (\cdot \cdot \cdot q_i + \cdot \cdot \cdot i) \Phi_N((\cdot \cdot \cdot q_i)p) + (\cdot \cdot \cdot q_i - \cdot \cdot \cdot i) \Phi_N((\cdot \cdot \cdot q_i)p) = (\cdot \cdot \cdot \cdot \cdot q_i \cdot \cdot \cdot i) \Phi((\cdot \cdot \cdot q_i)q)
\]

\[\Leftrightarrow (n)^{\frac{1}{2}} \Phi(n)^{\frac{1}{2}} = \frac{1}{\Phi(n)^{\frac{1}{2}}} \]

\[
(n)^{\frac{1}{2}} \Phi(n)^{\frac{1}{2}} \prod_{k=1}^{N} \sinh 2b(u) \sim (n)q
\]

Yields normalizable eigenstates of \( T(u) \).

(\ref{1}) \quad \Phi(t) \Phi(q) \quad \Phi(q) \Phi(t)

\[\text{Ansatz} \quad \Rightarrow (n)^{\frac{1}{2}} \Phi(n)^{\frac{1}{2}} = \frac{1}{\Phi(n)^{\frac{1}{2}}} \]

Bytsko, J. T. (Properties (i)-(iii))

\[\text{Key observations:} \quad (n)^{\frac{1}{2}} \Phi(n)^{\frac{1}{2}} \prod_{k=1}^{N} \sinh 2b(u) \sim (n)q\]

\[\text{Main idea (Sklyanin):} \quad \text{Diagonalize } B(u), \text{ parametrize eigenvalues as} \quad (n)^{\frac{1}{2}} \Phi(n)^{\frac{1}{2}} \prod_{k=1}^{N} \sinh 2b(u) \sim (n)q\]
Task: Classify set $Q$ of solutions to the Baxter equation (i)

$$\begin{align*}
\text{if } \frac{\zeta}{\pi} < \left| (n) \Re \right| & \quad \infty \leftarrow |n| \text{ } \text{ for } \zeta^{n}N^{l-n} \alpha N^{l-1} \beta \\
\text{if } \frac{\zeta}{\pi} > \left| (n) \Im \right| & \quad \infty \leftarrow |n| \text{ } \text{ for } \zeta^{n}N^{l-n} \alpha N^{l+1} \beta
\end{align*}$$

\begin{align*}
\left\{ \begin{array}{c}
\beta\zeta^{n}N^{l-n} \alpha N^{l-1} \beta \\
(\beta \zeta^{n}N^{l-n} \alpha N^{l+1} \beta)
\end{array} \right\} \sim (n)^{ib} \quad (iii)
\end{align*}

where

$$d(u) = a(u) = 1 + \left( \frac{m}{4} \right)^{2} e^{b(2u + ib)}$$

\begin{align*}
(i) & \quad q_{t}(u) q_{t}(u) = a(u) N q_{t}(u) + (a(u) N q_{t}(u) + b(u) N q_{t}(u) + b(u) N q_{t}(u)) \\
(ii) & \quad q_{t}(u) \text{ is meromorphic in } \mathbb{C}, \text{ with poles in } C \\
(iii) & \quad q_{t}(u) \leq \pi \\
\end{align*}

\begin{align*}
\left| (n) \Re \right| & \quad \infty \leftarrow |n| \\
\left| (n) \Im \right| & \quad \infty \leftarrow |n|
\end{align*}

A function $t(n)$ is eigenvalue of the transfer-matrix $\mathcal{T}(n)$ if and only if there exists a function $q_{t}(u)$ which satisfies $(n)^{ib}$
Let \( \mathcal{Y} \) be the set of all functions which satisfy the integral equation

\[
\mathcal{Y}(\beta) = \mathcal{Z}(\beta) \mathcal{L} + \mathcal{Y}(\beta) \mathcal{M}
\]

and which have the properties

\[
\mathcal{Z}(\beta) > |(n \mp \beta) \mathcal{G}| \quad \forall n \in \mathbb{N}, \quad \frac{2\pi}{n} \equiv \frac{1}{n}
\]

such that \( \mathcal{L} \) is meromorphic with poles of maximal order

\[
\varepsilon \frac{2\pi}{n} \equiv \frac{1}{n}
\]

\[
\mathcal{Z}(\beta) \mathcal{L} + \mathcal{Y}(\beta) \mathcal{M}
\]

are for

\[
(n \mp \beta) \mathcal{G}
\]

\[
\begin{align*}
\mathcal{Z}(\beta) &= \mathcal{L} + \mathcal{Y}(\beta) \\
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\mathcal{Z}(\beta) &= \mathcal{L} + \mathcal{Y}(\beta)
\end{align*}
\]

The existence of complex numbers \( \beta = \text{a}, \ldots, \beta = \text{a} \) such that

\[
\mathcal{Z}(\beta) \mathcal{L} + \mathcal{Y}(\beta) \mathcal{M}
\]

are for

\[
(n \mp \beta) \mathcal{G}
\]

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\end{align*}
\]
Theorem 1. There is a one-to-one correspondence between the solutions $Y(\#) \in Y_M$ of the integral equations (I) and the elements $Q \in Q_M$.

Given $Q \in Q_M$, get $Y(\#)$:

\[
W(\#) + Y(\#) = Q(\#) + i2Q(\#)
\]

\[
Z = \{\#_1, \ldots, \#_M\} \text{ set of zeros of } Q(\#) \text{ within } S.
\]

Given $Y(\#) \in Y_M$, get $Q$:

\[
Q(\#) = \left( \frac{\pi}{2} - \theta \right) \mathcal{O} \left( \frac{\pi}{2} + \theta \right) \mathcal{O} = (\theta)\mathcal{X} + (\theta)\mathcal{M}
\]

\[
\mathcal{X} \in (\theta)\mathcal{X} \quad \mathbb{Q}
\]

\[
\frac{\pi}{2} - \theta \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O}
\]

\[
\text{as } W_{\mathcal{O}} \in \mathcal{O} \text{ get } \{ \mathcal{O} \text{ of } \text{zeros: } \# \in \mathbb{Q}, \# \in \mathbb{Q} \} \text{ set of zeros within } S.
\]

\[
\mathcal{O} \subseteq (\theta)\mathcal{X} \quad \mathbb{Q}
\]

\[
((\theta)\mathcal{X} + (\theta)\mathcal{M}) \mathcal{O} = (\theta)\mathcal{O}
\]
Main claim:
\[ \left( \frac{2\theta}{s} \right)^N \exp \left( \frac{2\theta s}{mR} \right) \equiv \frac{2N \exp \frac{2\theta s}{mR}}{\theta} \]
is kept constant.

\( N \), \( s \) such that \( N \to \infty \), \( s \to \infty \) such that

Continuum limit: Easy!
Main Claim:

The Hilbert space of the Sinh-Gordon model contains (is equal to) $\mathcal{H}_{\text{TBA}}$.

\[ H \cong \bigoplus_{n=0}^{\infty} \mathcal{H} \]

The spaces $\mathcal{H}_M$ have ONB labelled by $k \in \mathbb{Z}$, $k = (k_1, \cdots, k_M)$, where $k_1 > \cdots > k_M$; $k \in \mathbb{Z}$. The corresponding function $Q_k(\#)$ can be represented as

\[ Q_k(\#) = \int_{\mathbb{R}} \frac{1}{|\theta|} \log \left( \frac{\rho_2 - \rho_1}{\rho_1} \right) \frac{\mu Z}{\theta \rho} \left( \cosh \theta \right)^{\rho_2 - \rho_1} \left( \left( \frac{\rho_2}{\rho_1} \right)^{\frac{\rho_2 - \rho_1}{\theta}} \right) \]

with $\theta$ being defined as the unique solution to

\[ 0 = \left( \frac{\partial}{\partial \theta} \right)^{-1} \cosh \theta + \left( \frac{\partial}{\partial \theta} \right)^{-1} \sinh \theta \]
The energies are calculated from $Y_T$ as follows:

$$E_k = m \cosh \theta a \int_{\mathbb{R}} m \cosh \nu_0 \log(1 + Y_T(\theta)) \sum_{I=1}^{p} \frac{Z}{\rho} \cosh \nu_m - m \cosh \nu_0 \log(1 + Y_T(\theta))$$

(Generalizes work of A.I. B. Zamolodchikov, S. Lukyanov for the ground state)

Excted state TBA!
\[(\theta_q - \theta_p) S^{p \neq q}_{\text{1} \neq q} \prod_{\eta} a_{\eta}^{q} \in \mathcal{F} \]

\[\text{Particle picture:} \]

\[M: \text{number of particles, } \quad \theta_a: \text{rapidity of particle } a, \quad \eta: \text{quantized by asymptotic Bethe ansatz equations}.\]

\[e^{i R \theta_p} = \frac{m \sinh \theta_p}{\theta_p} \quad \text{m: number of particles}, \quad \theta_p: \text{rapidity of particle } p.\]

\[\text{Considering IR limit } R \rightarrow \infty, \text{ notice that } Y, Y^\dagger, X \rightarrow \infty, \text{ and } m \rightarrow \infty.\]

The IR limit } R \rightarrow \infty.
\[ (\mathcal{H}_{\Phi} - \Phi) \mathcal{O} = (\beta) \mathcal{P} \lambda + 1 \mathcal{O} \left( \frac{\zeta}{\omega} \rho + \beta - \nu \beta \right) \frac{\mathcal{H} \mu \zeta}{\beta \rho} \int \equiv \nu \Phi \]

where

\[ \prod_{i=\lambda}^{\nu \neq q} (\nu \beta - \nu \beta) S = (\nu \Phi + \nu d) \mathcal{H}_{\Phi} - \Phi \]

--- Connection with Liouville theory!

The UV limit \( \mathcal{R} \rightarrow 0 \):

**Effects of vacuum polarization**

For finite \( \mathcal{R} \):
There is something better than Bethe ansatz.

But don't forget:

Prediction: Bethe ansatz will fail for $\text{AdS}_5$ sigma model, finite $J$.

(see Kotikov, Lipatov, Rej, Staudacher, Velizhanin)

$\text{Gauge theory side: }$ "Wrapping Interactions":

- Counterparts of $e^{-\mu R}$-corrections in sinh-Gordon (Schafer-Nameki, Zamaklar, Zarembo)

For finite $J$: corrections $O(e^{-2\mu J})$ to String Bethe Ansatz.

Valid for $J \rightarrow \infty$, $J$: angular momentum on $S^5$.

Compare with "String Bethe Ansatz".

Radius of world-sheet cylinder in Gauge-fixed action.