The Geometry of the Quantum Hall Effect

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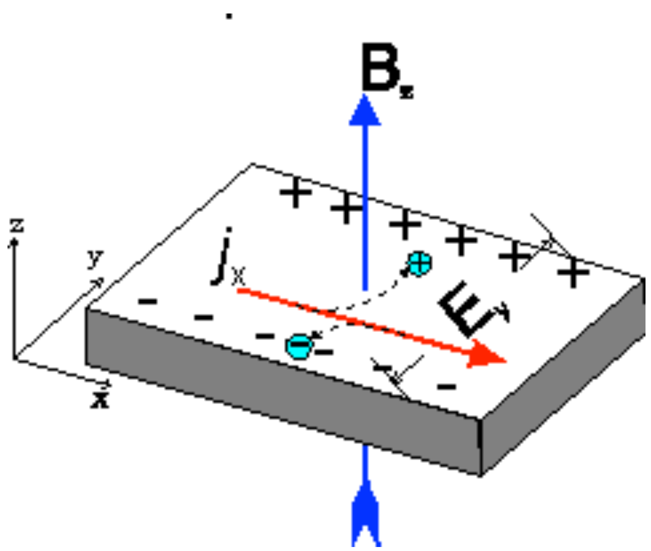
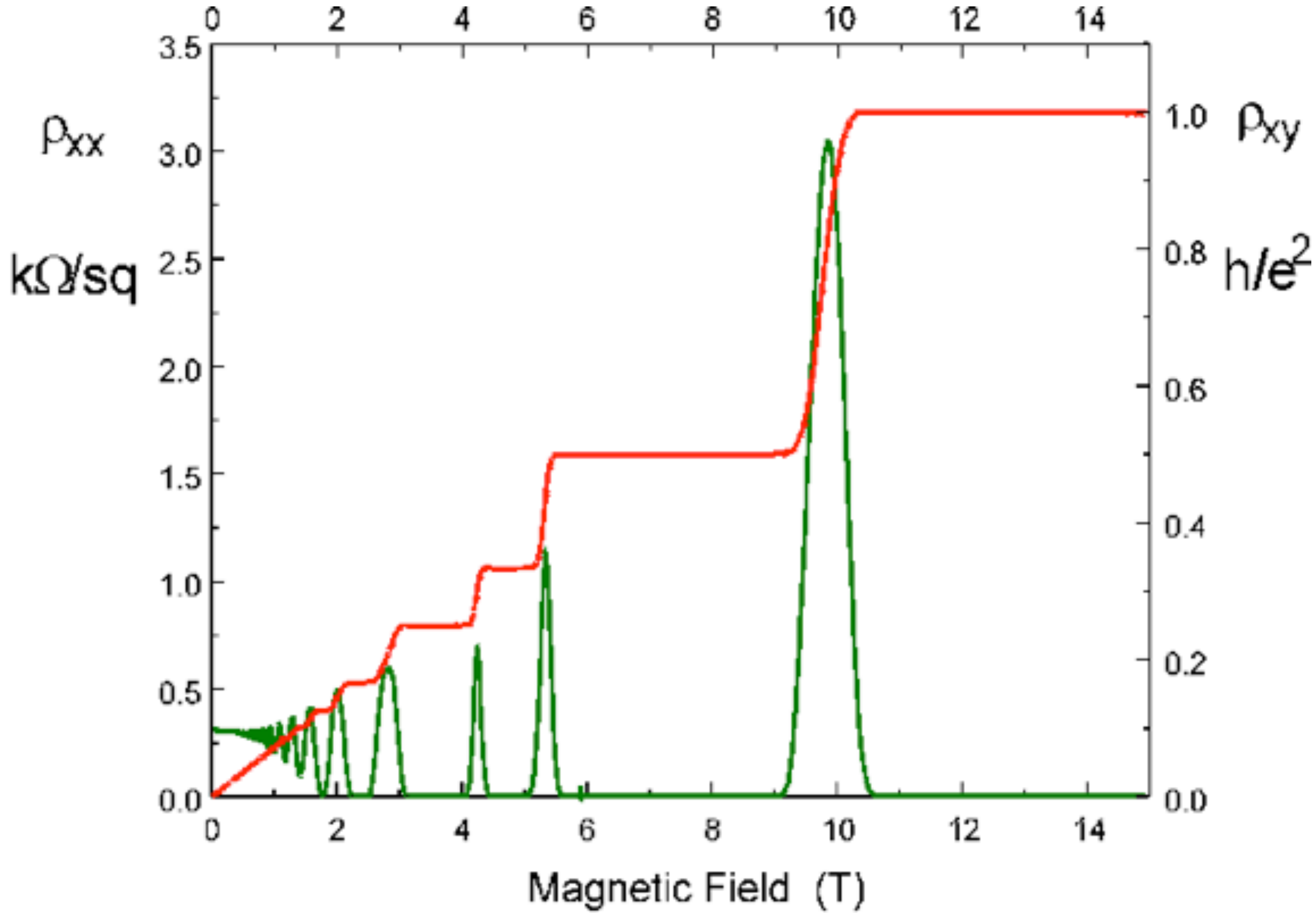
Refs:
Carlos Hoyos, DTS arXiv:1109.2651
DTS, M. Wingate cond-mat/0509786

Plan

- Review of quantum Hall physics
- Summary of results
- Method: Nonrelativistic diffeomorphism
- Chern-Simons and Wen-Zee terms
- Physical consequences

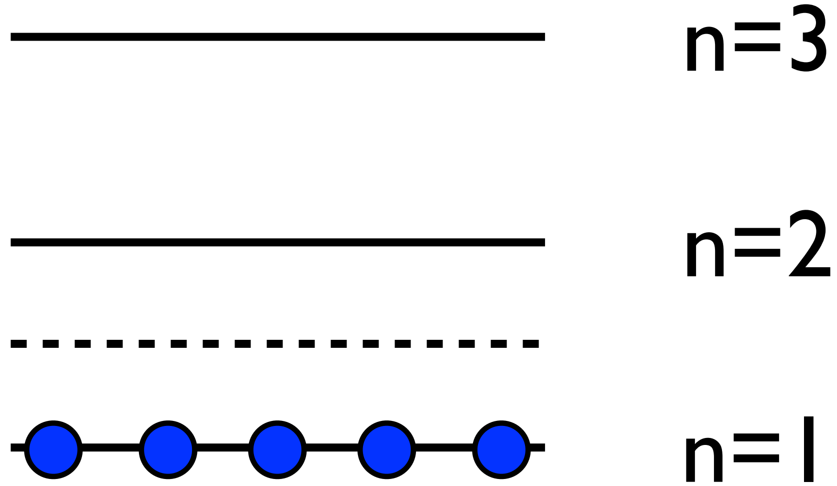
Quantum Hall effect





Integer quantum Hall state

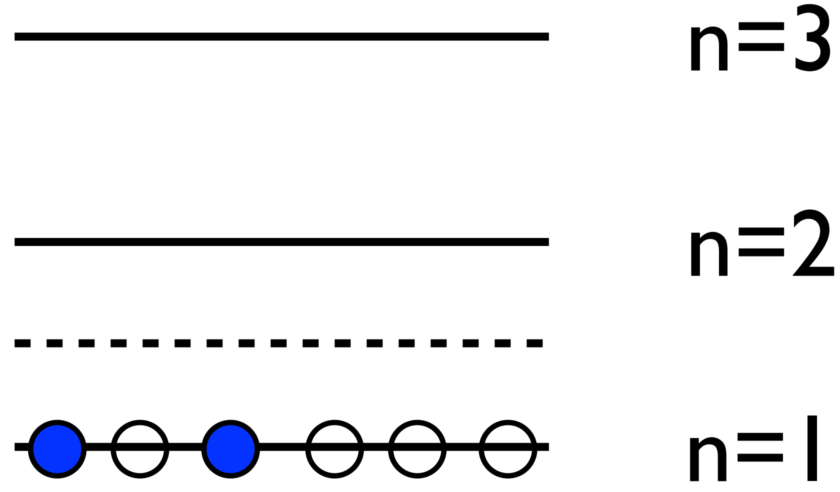
- simplest example: noninteracting electrons filling n Landau levels



degeneracy $\frac{B}{2\pi} \times \mathrm{Area}$

The state is completely gapped

Fractional quantum Hall state



Without interactions, ground state has huge degeneracy

Interactions somehow lift the degeneracy, make system gapped at particular values of the filling factor

$$\nu = \frac{n}{B/2\pi} \qquad \nu = \frac{1}{3}, \frac{1}{5}, \cdots$$

High B limit in FQHE

- Theoretically interesting limit:

$$\frac{e^2}{r} \ll \frac{B}{m} \qquad\qquad r = n^{-1/2} \sim B^{-1/2}$$

All interesting physics happens in the lowest Landau level

- Many properties captured by Laughlin's trial wave function

$$\psi(z) = \prod_{\langle ij \rangle} (z_i - z_j)^3 \prod_i e^{-|z_i|^2/4\ell^2}$$

Effective field theory

- Effective field theory: captures low-energy dynamics
- What are the low-energy degrees of freedom of a quantum Hall state?
  - there are none (in the bulk): energy gap
  - Thus the effective Lagrangian is polynomial over external fields

(we will consider only clean systems)

Chern-Simons action

- The orthodox point of view is that all universal information about transport is encoded in the Chern-Simons action:

$$S = \frac{\nu}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

encodes Hall conductivity

$$J^\mu = \frac{\delta S}{\delta A_\mu} \qquad \rho = \frac{\nu}{2\pi} B$$

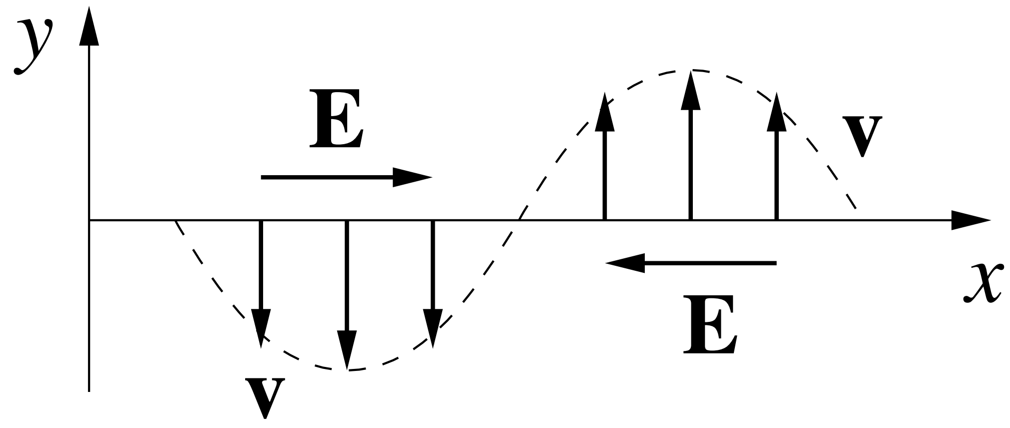
$$J_y = \sigma_{xy} E_x \qquad \sigma_{xy} = \frac{\nu}{2\pi}\frac{e^2}{\hbar}$$

Universality beyond CS

Higher-derivatives corrections: of dynamical, not topological nature, hence non universal?

We will show that there is universality beyond the CS action

$\sigma_{xy}(q)$



$$E_x = E\,\mathrm{e}^{iqx}$$

$$j_y = \sigma_{xy}(q) E_x$$

$$\sigma_{xy}(q) = \frac{\nu}{2\pi}\left[1 + \left(\frac{\mathcal{S}}{4} - 1\right)(q\ell)^2\right] + O(q^4\ell^4)$$

$\mathcal{S}$ is the "shift": topological property of a state

$\mathcal{S} = 1/\nu$ for Laughlin's state

exact result, insensitive of interactions

Symmetries of NR theory

DTS, M.Wingate 2006

Microscopic theory

$$S_0 = \int \mathrm{d}t\, \mathrm{d}^2x\, \sqrt{g} \Big[\frac{i}{2} \psi^\dagger \overleftrightarrow{D}_t \psi - \frac{g^{ij}}{2m} D_i \psi^\dagger D_j \psi \Big] \qquad D_\mu \psi \equiv (\partial_\mu - i A_\mu)\psi$$

Gauge invariance: $\psi \to e^{i\alpha}\psi \qquad A_\mu \to A_\mu + \partial_\mu \alpha$

General coordinate invariance:

$$\delta\psi = -\xi^k \partial_k \psi \equiv \mathcal{L}_\xi \psi$$

$$\delta A_0 = \xi^k \partial_k A_0 \equiv \mathcal{L}_\xi A_0$$

$$\delta A_i = -\xi^k \partial_k A_i - A_k \partial_i \xi^k \equiv \mathcal{L}_\xi A_i$$

$$\delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{kj} \partial_i \xi^k - g_{ik} \partial_j \xi^k \equiv \mathcal{L}_\xi g_{ij}$$

Here ξ is time independent: ξ=ξ(**x**)

NR diffeomorphism

- These transformations can be generalized to be time-dependent: $\xi=\xi(t,\mathbf{x})$

$$\delta\psi = -\mathcal{L}_\xi \psi$$

$$\delta A_0 = -\mathcal{L}_\xi A_0 - A_k \dot{\xi}^k$$

$$\delta A_i = -\mathcal{L}_\xi A_i - m g_{ik} \dot{\xi}^k$$

$$\delta g_{ij} = -\mathcal{L}_\xi g_{ij}$$

Galilean transformations: special case $\xi^i=v^i t$

Where does it come from

Start with complex scalar field

$$S = -\int \mathrm{d}x \, \sqrt{-g} \, (g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + \phi^* \phi)$$

Take nonrelativistic limit:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dfrac{2A_0}{mc^2} & \dfrac{A_i}{mc} \\ \dfrac{A_i}{mc} & g_{ij} \end{pmatrix}$$

$$\phi = e^{-imcx^0} \frac{\psi}{\sqrt{2mc}}$$

$$S = \int \mathrm{d}t \, \mathrm{d}\mathbf{x} \, \sqrt{g} \left[\frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_t \psi + A_0 \psi^\dagger \psi - \frac{g^{ij}}{2m} (\partial_i \psi^\dagger + i A_i \psi^\dagger)(\partial_j \psi - i A_j \psi) \right].$$

Relativistic diffeomorphism

$$x^\mu \to x^\mu + \xi^\mu$$

μ =0: gauge transform

$$\phi = e^{-imcx^0}\frac{\psi}{\sqrt{2mc}}$$

μ =i: general coordinate transformations

Interactions

- Interactions can be introduced that preserve nonrelativistic diffeomorphism
  - interactions mediated by fields
- For example, Yukawa interactions

$$S = S_0 + \int d^3x \, \sqrt{g}\phi\psi^\dagger\psi - \int d^3x \sqrt{g} \left(g^{ij}\partial_i\phi\partial_j\phi + M^2\phi^2 \right)$$

$$\delta\phi = -\xi^k\partial_k\phi$$

Is CS action invariant?

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$$\delta S_{\mathrm{CS}} = \frac{\nu m}{2\pi}\int \mathrm{d}t\, \mathrm{d}^2x\, \epsilon^{ij} E_i g_{jk} \dot{\xi}^k$$

Is CS action invariant?

- CS action is gauge invariant
- CS action is Galilean invariant
- CS action is *not* diffeomorphism invariant

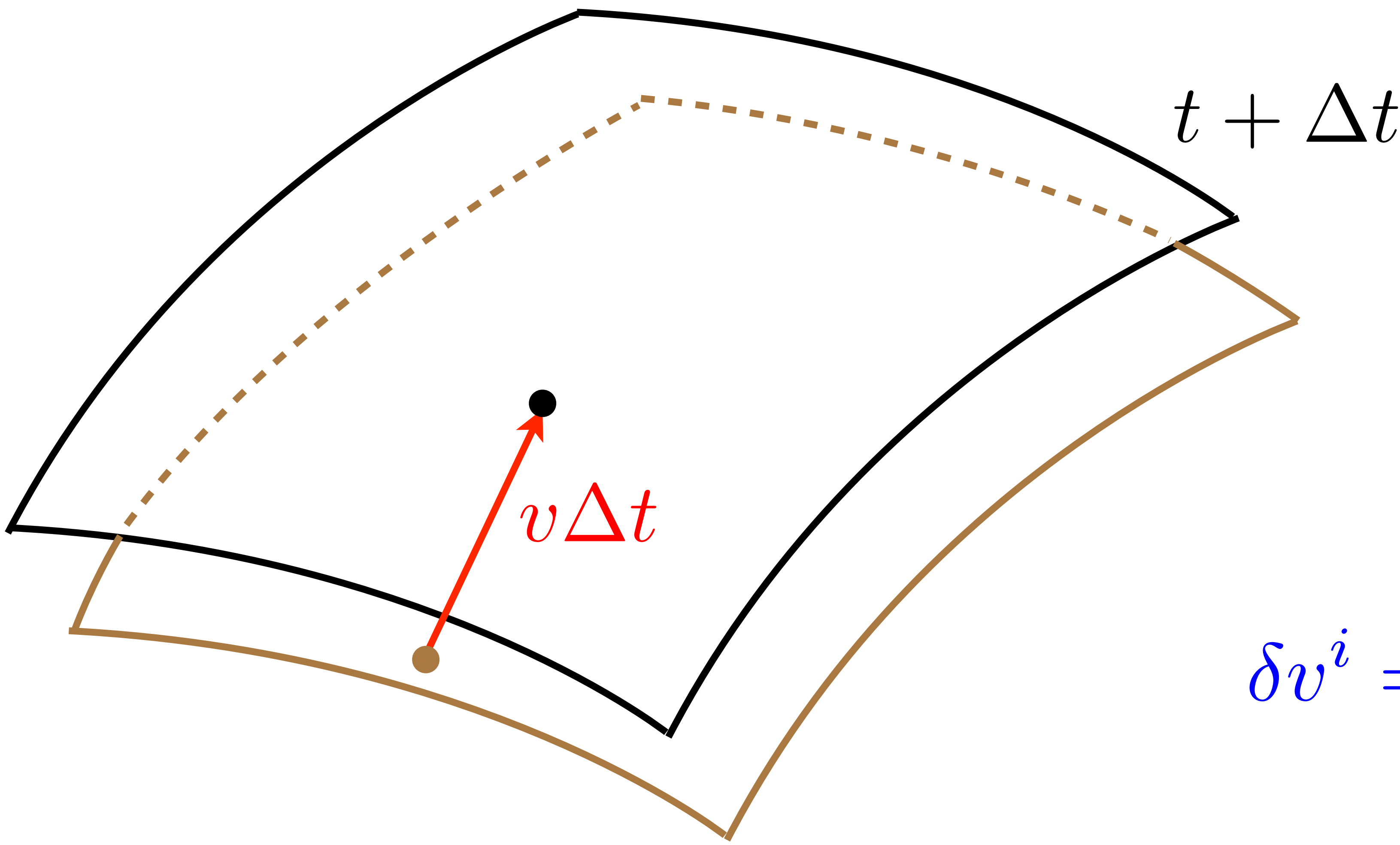
$$\delta S_{\mathrm{CS}} = \frac{\nu m}{2\pi} \int \mathrm{d}t\, \mathrm{d}^2x\, \epsilon^{ij} E_i g_{jk} \dot{\xi}^k$$

A_μ does not transform like a one-form

More on geometry

- System does not live in a 3D Riemann space
- 2D Riemann manifold at any time slice
    - can parallel transport along equal-time slices, but not between different times

Velocity vector v



$$\delta v^i = -[\xi,\, v]^i + \dot{\xi}^i$$

Can use v to transform objects from one time slice to another

Different v's differ by a 2D vector: $\delta(\tilde{v}^i - v^i) = -[\xi,\, \tilde{v} - v]^i$

Connection

with v one can define a connection:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\hat{g}^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

$$\hat{g}^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} \qquad\qquad g_{\mu\nu} = \begin{pmatrix} v^2 & -v_j \\ -v_i & g^{ij} \end{pmatrix}$$

$$\Gamma^{j}_{0i} = \frac{1}{2}g^{jk}(\partial_k v_i - \partial_i v_k + \dot{g}_{ik})$$

$$\Gamma^{j}_{00} = -g^{jk}\left(\dot{v}_k + \frac{1}{2}\partial_k v^2\right)$$

Improved gauge potentials

- With v one can construct a gauge potential that transforms as a one-form

$$\tilde{A}_i = A_i + m v_i$$

$$\tilde{A}_0 = A_0 - \frac{m v^2}{2}$$

$$\delta \tilde{A}_\mu = -\xi^k \partial_k \tilde{A}_\mu - \tilde{A}_k \partial_\mu \xi^k$$

$$\tilde{B} = \partial_1 \tilde{A}_2 - \partial_2 \tilde{A}_1 \qquad \tilde{E}_i = \partial_i \tilde{A}_0 - \partial_0 \tilde{A}_i$$

$\dfrac{\epsilon^{ij} \tilde{E}_j}{\tilde{B}}$ transforms like v^i

v as fluid velocity

- We can thus require

$$v^i = \frac{\epsilon^{ij}\tilde{E}_j}{\tilde{B}}$$

This is a nonlinear equation for v

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

hydrodynamic equation of an forced ideal fluid without pressure

Can be solved by iteration: $v^i = \frac{\epsilon^{ij}E_j}{B} + \cdots$

CS term corrected

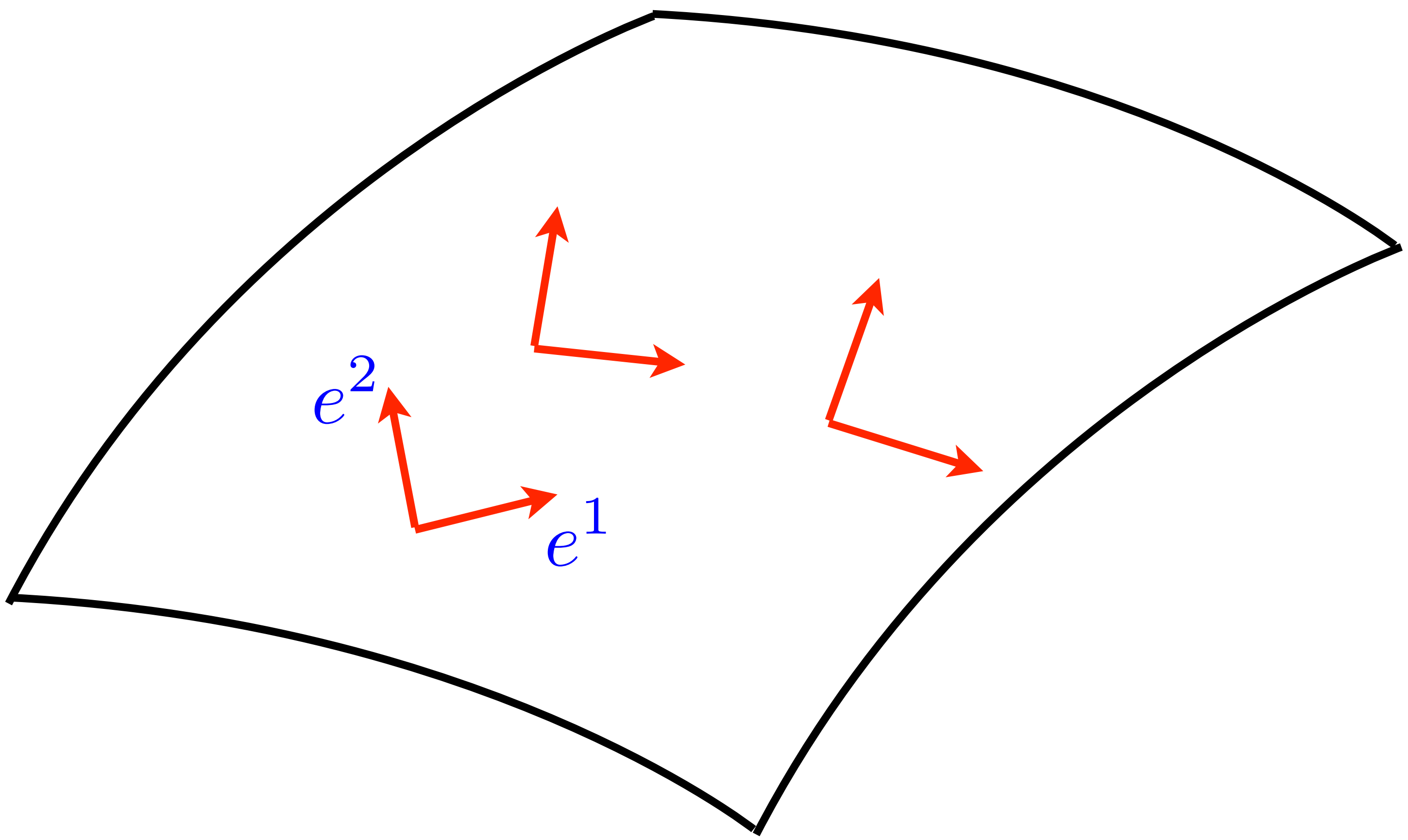
$$S = \frac{\nu}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\tilde{A}_\mu \partial_\nu \tilde{A}_\lambda$$

$$= S_{\rm CS} + \frac{\nu}{4\pi}\frac{mE^2}{B} + \cdots$$

fixed coefficient
Kohn's theorem

but it is not the only Chern-Simons term

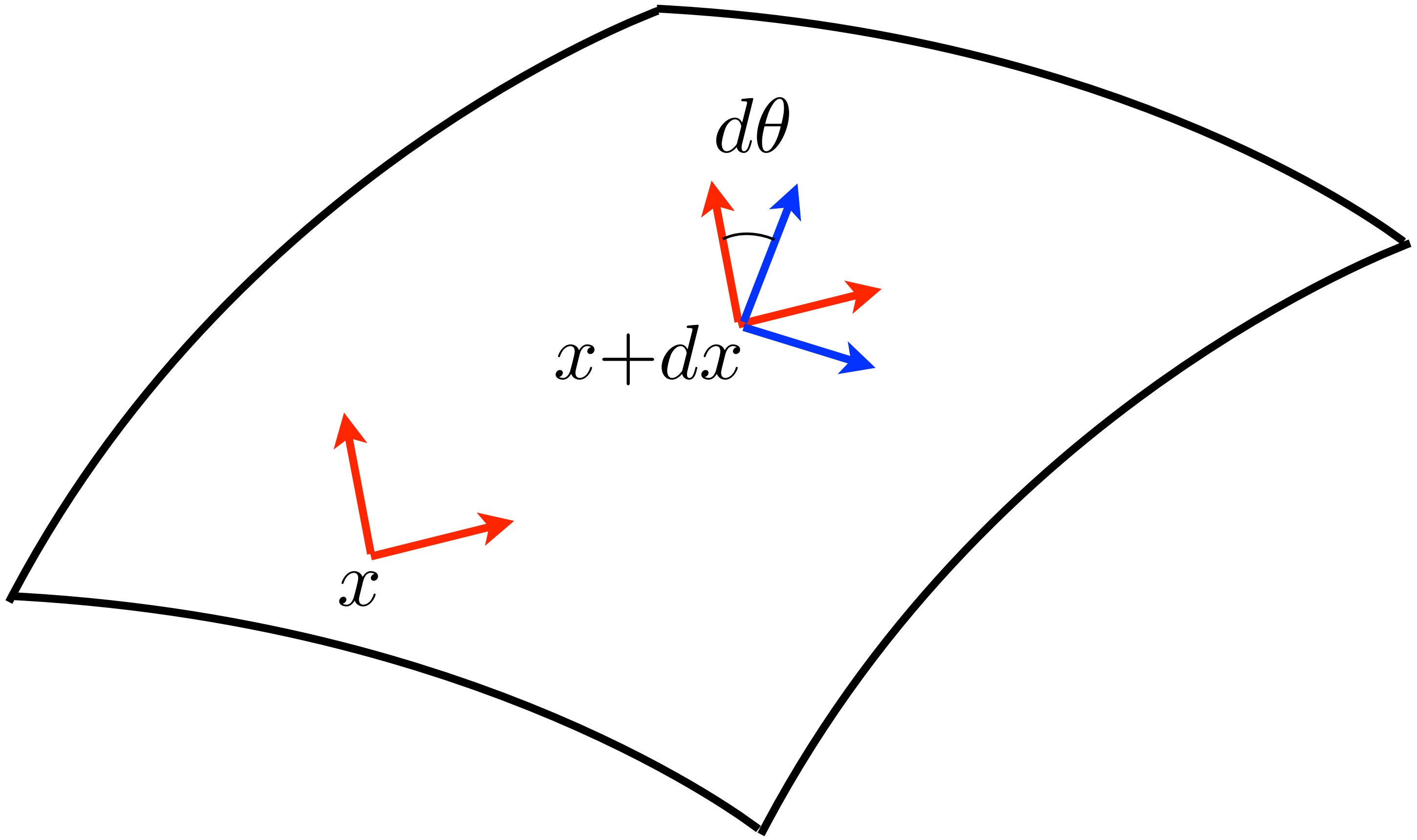
Spin connection



Choose orthonormal basis at each point $e_i^a(x)$

$a = 1, 2$

The spin connection



parallel transporting
from x to x+dx

$$d\theta = \omega_i(x)dx^i$$

$$\omega_i = \frac{1}{2}\epsilon^{ab}e^{aj}\nabla_i e^b_j$$

Spin connection: time component

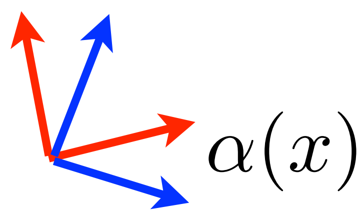
$$\omega_0 = \frac{1}{2}\epsilon^{ab} e^{ai} \nabla_0 e_i^b$$

$$= \frac{1}{2}\epsilon^{ab} e^{ai} \partial_t e_i^b + \frac{1}{2}\epsilon^{ij} \partial_i v_j$$

nonzero
in flat space

Spin connection as U(1) gauge field

- Under local rotation of the vielbein



$$\omega_\mu \to \omega_\mu - \partial_\mu \alpha$$

curvature is flux of ω

$$\partial_1 \omega_2 - \partial_2 \omega_1 = \frac{1}{2}\sqrt{g}\, R$$

ω_μ transforms like a (2+1)D one-form

Two more CS terms

$\epsilon^{\mu\nu\lambda} \tilde{A}_\mu \partial_\nu \omega_\lambda$ shift (Wen-Zee)

$\epsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu \omega_\lambda$

Wen-Zee shift

$$\frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu\omega_\lambda = \frac{\kappa}{4\pi}\sqrt{g}\,A_0 R + \cdots$$

Total particle number on a manifold:

$$Q = \int \mathrm{d}^2x\,\sqrt{g}\,j^0 = \int \mathrm{d}^2x\,\sqrt{g}\left(\frac{\nu}{2\pi}B + \frac{\kappa}{4\pi}R\right) = \nu N_\phi + \kappa\chi$$

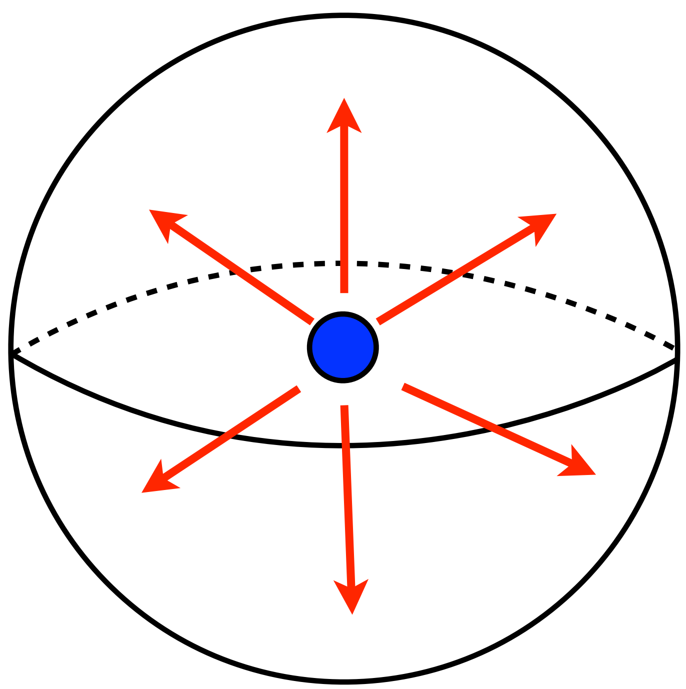
On a sphere: $Q = \nu(N_\phi + \mathcal{S}), \quad \mathcal{S} = \dfrac{2\kappa}{\nu}$

'shift'

IQH states: ν=n, κ=n^2/2

Laughlin's states: ν=1/n, κ=1/2

Shift for IQH states



$N_\phi + 5$

$N_\phi + 3$

$N_\phi + 1$

$$Q = nN_\phi + n^2 = n(N_\phi + n)$$

Back to flat space

$$\frac{\nu}{4\pi}\tilde{A}\,\partial\tilde{A} \qquad \frac{\nu}{4\pi}\left(A\,\partial A+\frac{mE^2}{B}\right)$$

$$\frac{\kappa}{2\pi}\tilde{A}\partial\omega \qquad \frac{\kappa}{4\pi B}\mathbf{E}\cdot\boldsymbol{\nabla}B$$

$$-\epsilon(\tilde{B}) \qquad -\epsilon(B)-\frac{m}{B}\epsilon''(B)\mathbf{E}\cdot\boldsymbol{\nabla}B$$

Back to flat space

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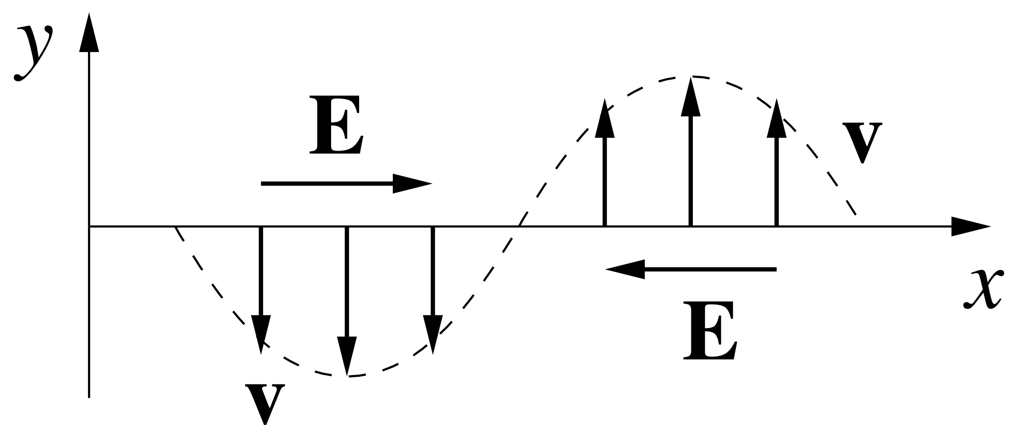
q^2 correction to Hall conductivity

$$E\nabla B \sim \nabla A_0 \nabla B \sim q^2 A_0 B$$

$\sigma_{xy}(q)$



$$E_x = E\,\mathrm{e}^{iqx}$$

$$j_y = \sigma_{xy}(q) E_x$$

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + C_2 (q\ell)^2 + \mathcal{O}(q^4 \ell^4)$$

$$C_2 = \frac{\mathcal{S}}{4} - \frac{2\pi}{\nu}\,\frac{\ell^2}{\hbar\omega_c} B^2 \epsilon''(B)$$

Hall viscosity from WZ term

$$S_{\rm WZ} = -\frac{\kappa B}{16\pi}\epsilon^{ij} h_{ik}\partial_t h_{jk} + \cdots$$

$$\langle T_{xy}(T_{xx} - T_{yy})\rangle \sim i\eta^a \omega$$

$$\eta^a = \frac{\kappa B}{4\pi} = \frac{1}{4}\mathcal{S} n$$

derived by N.Read
previously

What is Hall viscosity?

Standard fluid dynamics: $\partial_t \rho + \partial_i j^i = 0$ continuity eq.

$\partial_t j^i + \partial_j T^{ij} = 0$ Navier-Stokes eq.

$$j^i = \rho v^i$$

$$T^{ij} = \rho v^i v^j + P\delta^{ij} - \eta V_{ij} \qquad V_{ij} = \frac{1}{2}(\partial_i v^j + \partial_j v^i)$$

In 2 spatial dimensions, it is possible to write

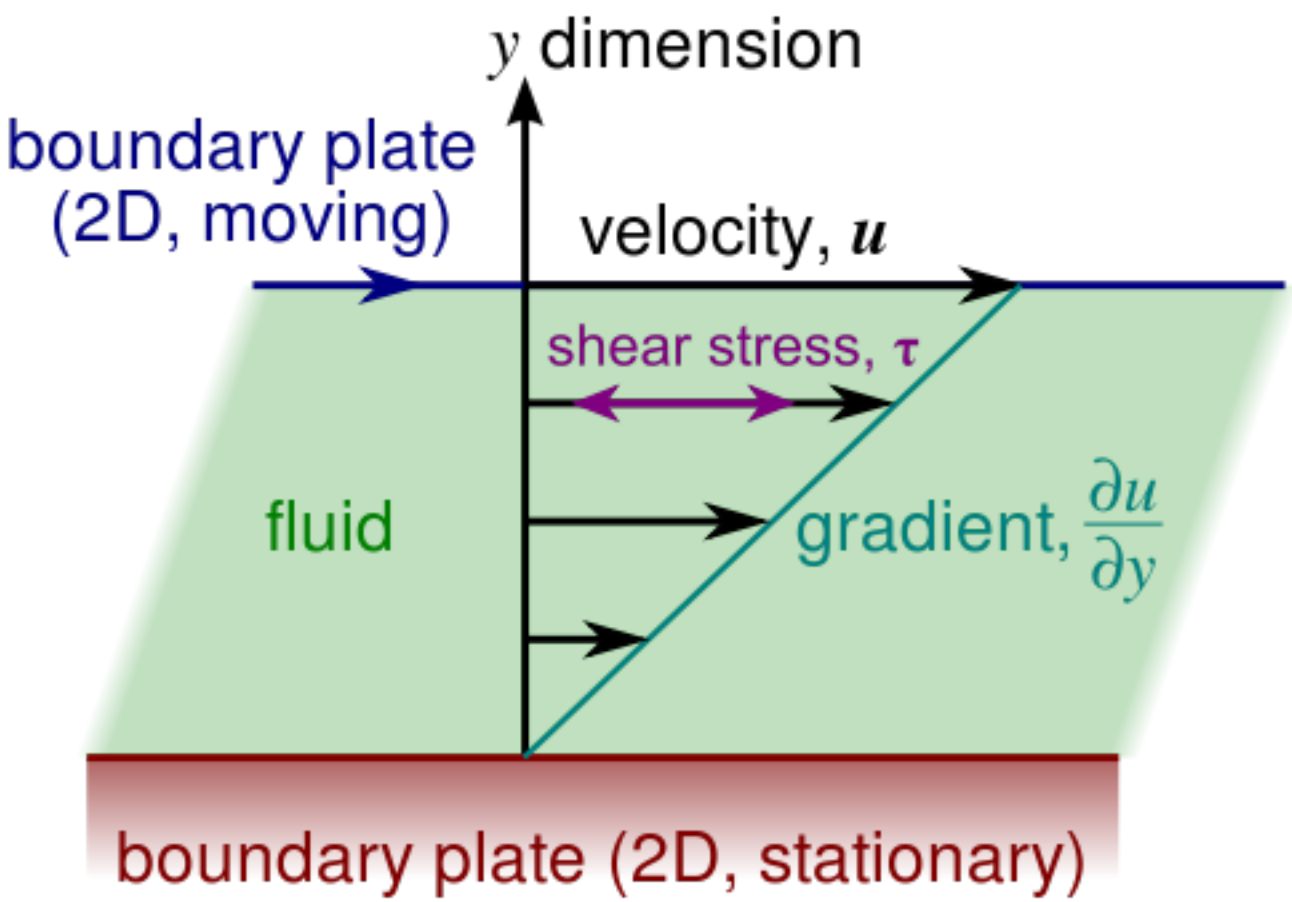
$$T^{ij} = \cdots - \eta_H(\epsilon^{ik}V^{kj} + \epsilon^{jk}V^{ki})$$

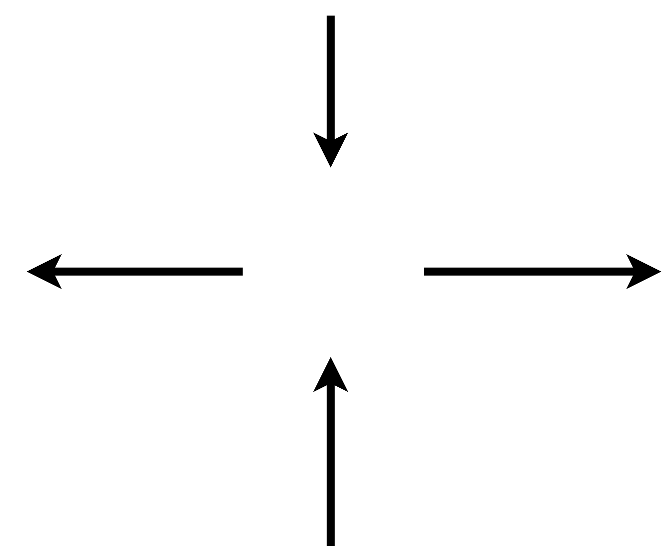
breaks parity

dissipationless

Hall viscosity (Avron Seiler Zograf)

Hall viscosity in picture

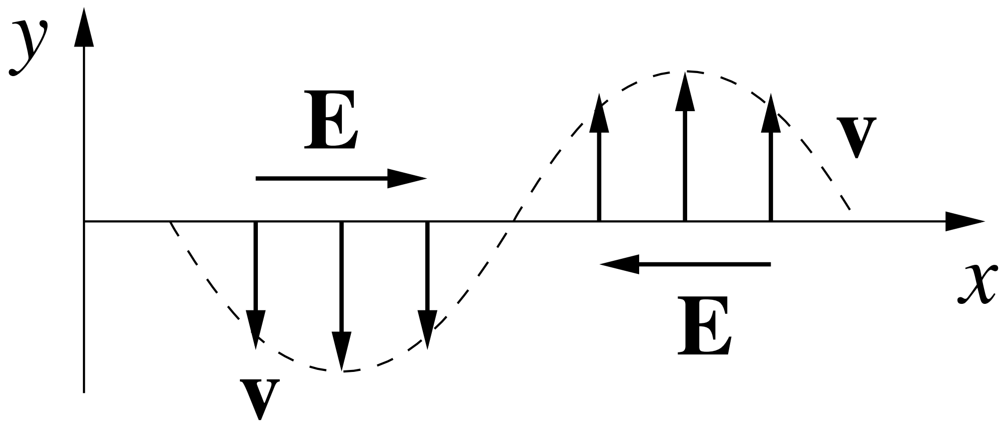




Hall shear stress

Physical interpretation

- First term: Hall viscosity



$$\partial_x v_y + \cancel{\partial_y v_x} \neq 0$$

$$T_{xx} = T_{xx}(x) \neq 0$$

additional force $F_x \sim \partial_x T_{xx}$

Hall effect: additional contribution to v_y

Physical interpretation (II)

- 2nd term: more complicated interpretation

Fluid has nonzero angular velocity

$$\Omega(x) = \frac{1}{2}\partial_x v_y = -\frac{cE_x'(x)}{2B}$$

$$\delta B = 2mc\Omega/e$$

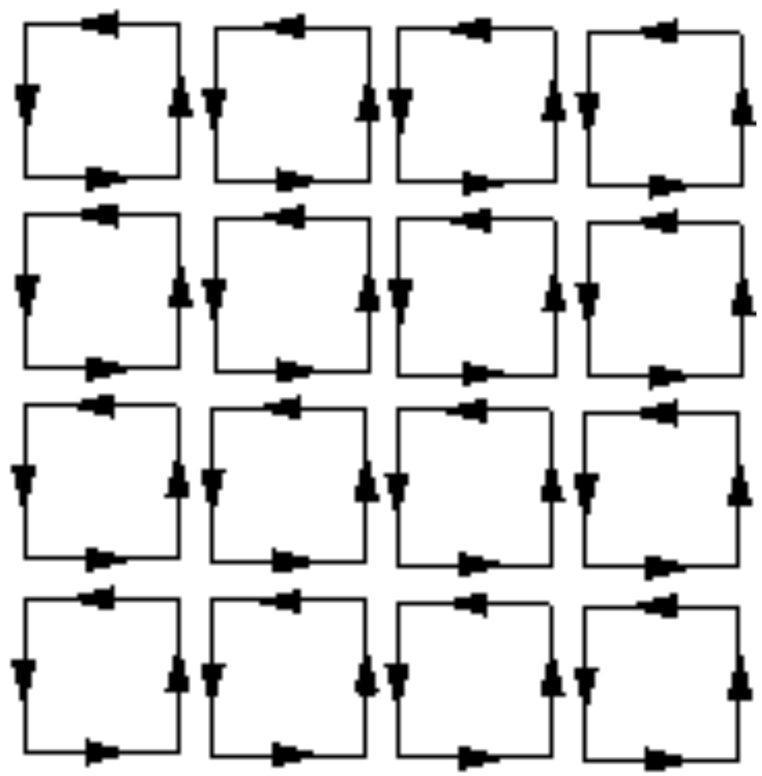
Coriolis=Lorentz

Hall fluid is diamagnetic: $d\epsilon = -M dB$

M is spatially dependent M=M(x)

Extra contribution to current $\mathbf{j} = c\,\hat{\mathbf{z}} \times \nabla M$

Current ~ gradient of magnetization



$$\mathbf{j} = c\,\hat{\mathbf{z}} \times \nabla M$$

High B limit

- In the limit of high magnetic field: ϵ(B) known: free fermions
  - n Landau levels for IQH states
  - first Landau level for FQH states with ν<1
- Wen-Zee shift is known

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 - \frac{3n}{4}(q\ell)^2 + \mathcal{O}(q^4\ell^4) \qquad \nu = n$$

matches explicit calculations

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + \frac{2n-3}{4}(q\ell)^2 + \mathcal{O}(q^4\ell^4), \quad \nu = \frac{1}{2n+1}$$

exact

Conclusion

- Quantum Hall states live in a special type of geometric structure: global time, (g_{ij}, v^i)
- Symmetry determines the q^2 correction to Hall conductivity, related to Hall viscosity
- Open questions:
  - edge states?
  - Lowest Landau level: additional symmetries?
  - AdS/CFT realizations?

Power counting

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  - magnetic length $\ell \sim B^{-1/2}$
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- Assume $\partial_i \sim \varepsilon B^{1/2}$, $\partial_0 \sim \varepsilon^2 B/m$ $\quad$ ($z=2$)

Power counting

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- Assume $\partial_i \sim \varepsilon B^{1/2}$, $\partial_0 \sim \varepsilon^2 B/m$ (z=2)

$$\delta A_i \sim \frac{B^{1/2}}{\epsilon}, \quad \delta A_0 \sim \frac{B}{m}, \quad \delta g_{ij} \sim 1$$

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Slowly varying, nonlinear external fields

$$\delta B \sim B, \quad \delta A_0 \sim \mu, \quad \delta g_{ij} \sim 1$$

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Slowly varying, nonlinear external fields

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$$E \sim \epsilon \frac{B^{3/2}}{m}$$