

Cold atoms and AdS/CFT

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Plan

- History/motivation
 - BCS/BEC crossover
 - Unitarity regime
- Schrödinger symmetry
- Geometric realization of Schrödinger symmetry
- Transport coefficients of holographic nonrelativistic fluids
- Conclusion

Collaborators: M. Rangamani, S. Ross, E. Thompson

Refs.: D.T. Son, 0804.3972 (PRD 2008)

also Balasubramanian and McGreevy, 0804.4053 (PRL 2008)

Rangamani, Ross, DTS, Thompson, arXiv:0811.2049

Part of the results obtained in collaboration with Yusuke Nishida

BCS mechanism

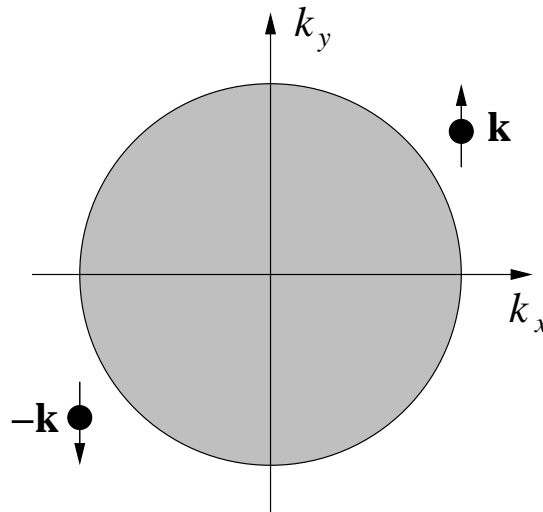
Explains superconductivity of metals

Consider a gas of spin-1/2 fermions ψ_a , $a = \uparrow, \downarrow$

No interactions: Fermi sphere: states with $k < k_F$ filled

Cooper phenomenon: any attractive interaction leads to condensation

$$\langle \psi_{\uparrow} \psi_{\downarrow} \rangle \neq 0$$



BCS/BEC crossover

Critical temperature:

$$T_c \sim \epsilon_F e^{-1/g}, \quad g \ll 1$$

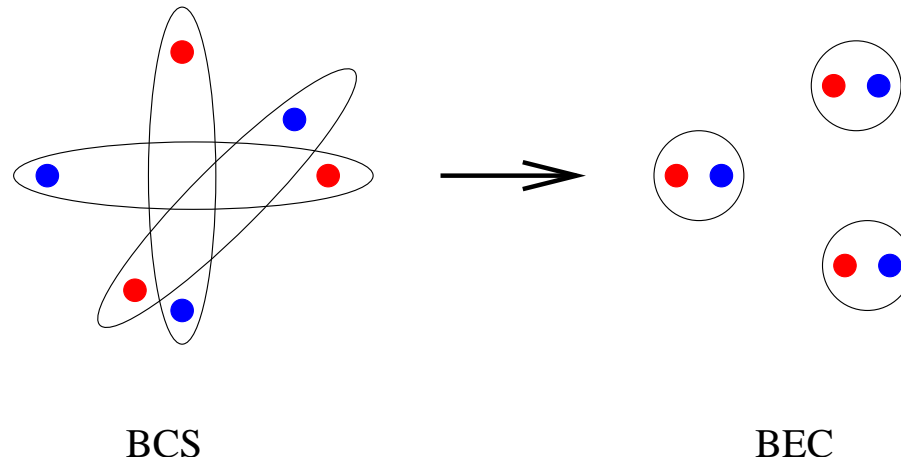
Leggett (1980): If one increases the interaction strength g , how large can one make T_c/ϵ_F ?

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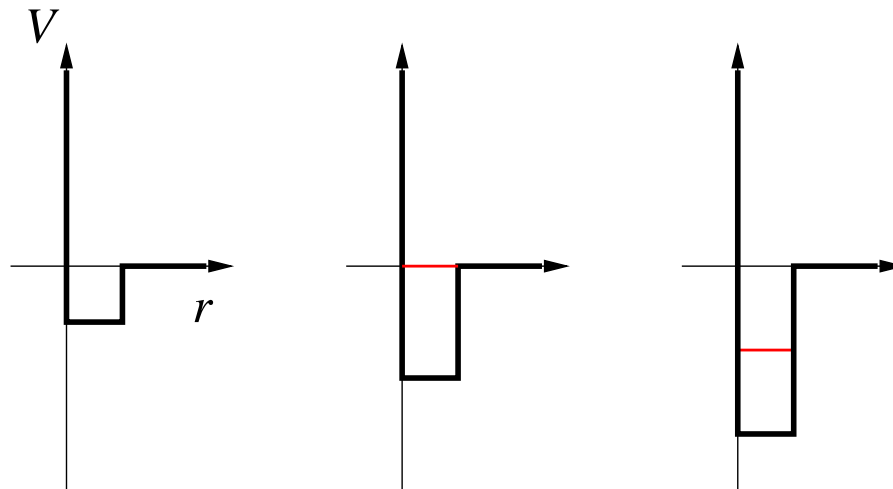


Strong attraction: leads to bound states, which form a dilute Bose gas
 $T_c =$ temperature of Bose-Einstein condensation $\approx 0.22\epsilon_F$.

Unitarity regime

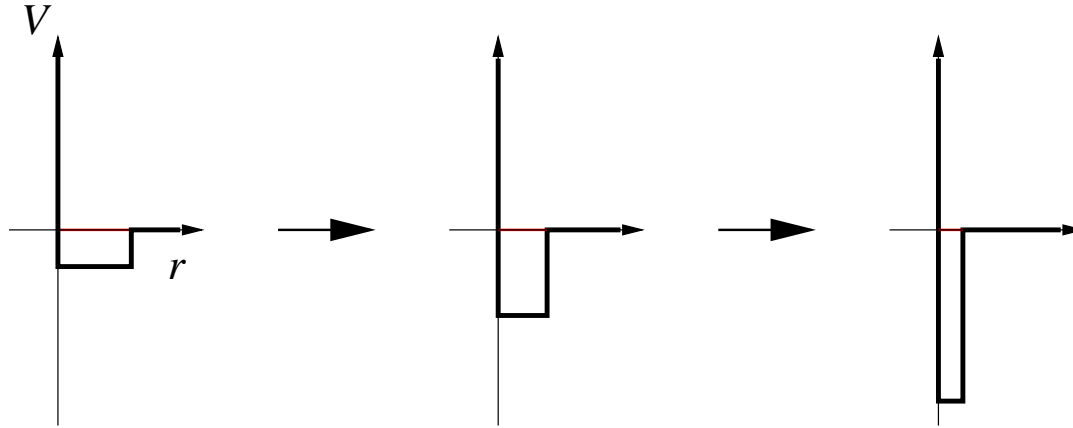
Consider in more detail the process of going from BCS to BEC
Assume a square-well potential, with fixed but small range r_0 .

- $V_0 < 1/mr_0^2$: no bound state
- $V_0 = 1/mr_0^2$: one bound state appears, at first with zero energy
- $V_0 > 1/mr_0^2$: at least one bound state



Unitarity regime (II)

Unitarity regime: take $r_0 \rightarrow 0$, keeping one bound state at zero energy.



In this limit: no intrinsic scale associated with the potential

In the language of scattering theory: infinite scattering length $a \rightarrow \infty$

s -wave scattering cross section saturates unitarity

Boundary condition interpretation

Unitarity: taking Hamiltonian to be free:

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m}$$

but imposing nontrivial boundary condition on the wavefunction:

$$\Psi(\underbrace{\mathbf{x}_1, \mathbf{x}_2, \dots}_{\text{spin-up}}, \underbrace{\mathbf{y}_1, \mathbf{y}_2, \dots}_{\text{spin=down}})$$

When $|\mathbf{x}_i - \mathbf{y}_j| \rightarrow 0$:

$$\Psi \rightarrow \frac{C}{|\mathbf{x}_i - \mathbf{y}_j|} + 0 \times |\mathbf{x}_i - \mathbf{y}_j|^0 + O(|\mathbf{x}_i - \mathbf{y}_j|)$$

Free gas corresponds to

$$\Psi \rightarrow \frac{0}{|\mathbf{x}_i - \mathbf{y}_j|} + C + O(|\mathbf{x}_i - \mathbf{y}_j|)$$

Field theory interpretation

Consider the following model

Sachdev, Nikolic; Nishida, DTS

$$S = \int dt d^d x \left(i\psi^\dagger \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 - c_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right)$$

Dimensional analysis:

$$[t] = -2, \quad [x] = -1, \quad [\psi] = \frac{d}{2}, \quad [c_0] = 2 - d$$

Contact interaction is irrelevant at $d > 2$

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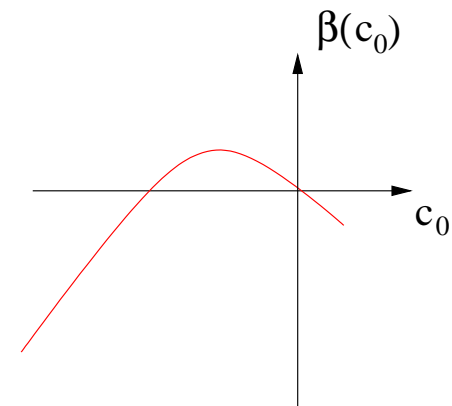
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RG equation in $d = 2 + \epsilon$:

$$\frac{\partial c_0}{\partial s} = -\epsilon c_0 - \frac{c_0^2}{2\pi}$$

Two fixed points:

- $c_0 = 0$: trivial, noninteracting
- $c_0 = -2\pi\epsilon$: unitarity regime



Field theory in $d = 4 - \epsilon$ dimensions

Sachdev, Nikolic; Nishida, DTS

$$S = \int dt d^d x \left(i\psi^\dagger \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 - g\phi\psi_\uparrow^\dagger \psi_\downarrow^\dagger - g\phi^* \psi_\downarrow \psi_\uparrow + i\phi^* \partial_t \phi - \frac{1}{4m} |\nabla \phi|^2 + C\phi^* \phi \right)$$

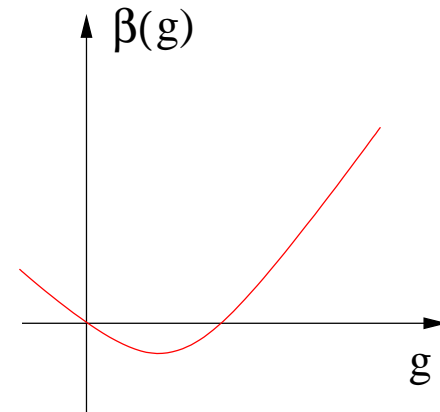
C finely tuned to criticality

Dimensions: $[g] = \frac{1}{2}(4 - d) = \frac{1}{2}\epsilon$

RG equation for g :

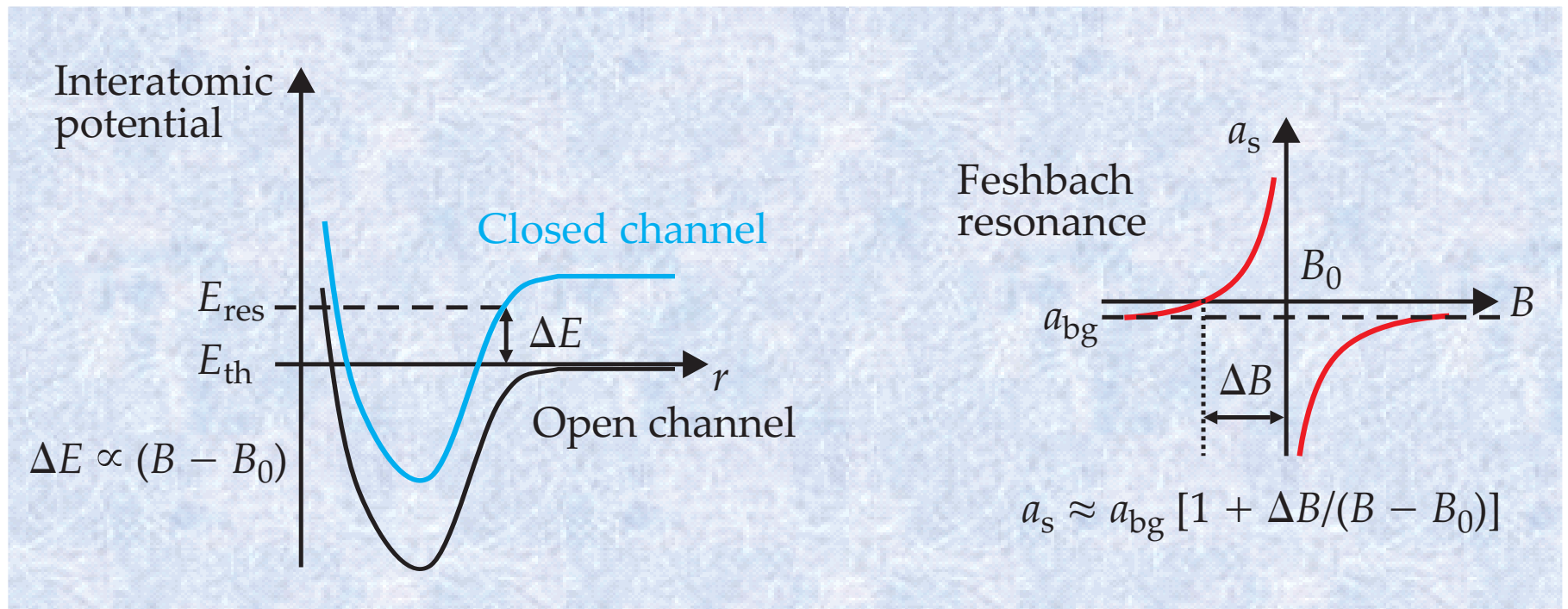
$$\frac{\partial g}{\partial \ln \mu} = -\frac{\epsilon}{2}g + \frac{g^3}{16\pi^2}$$

Fixed point at $g^2 = 8\pi^2\epsilon$



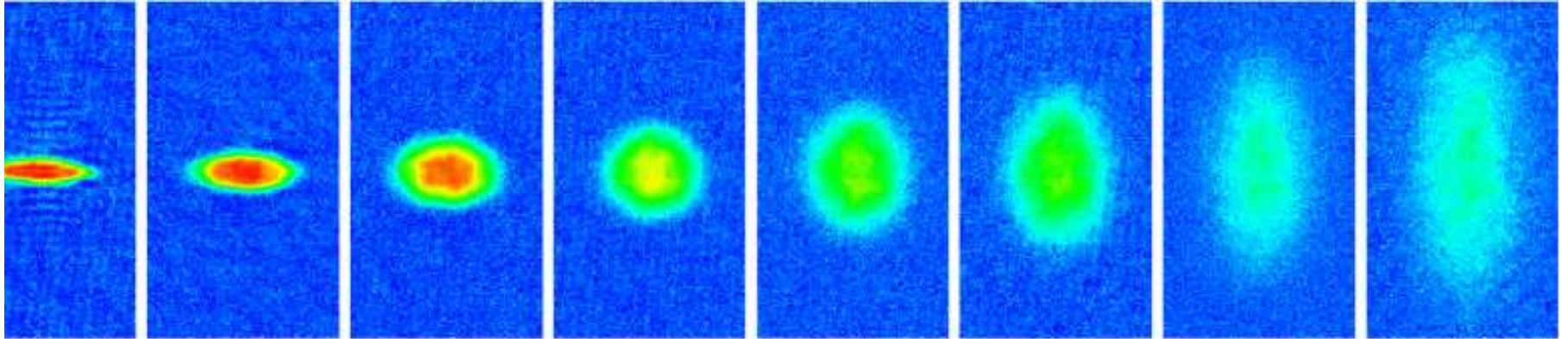
Examples of systems near unitarity

- Neutrons: $a = -20 \text{ fm}$, $|a| \gg 1 \text{ fm}$
- Trapped atom gases, with scattering length a controlled by magnetic field

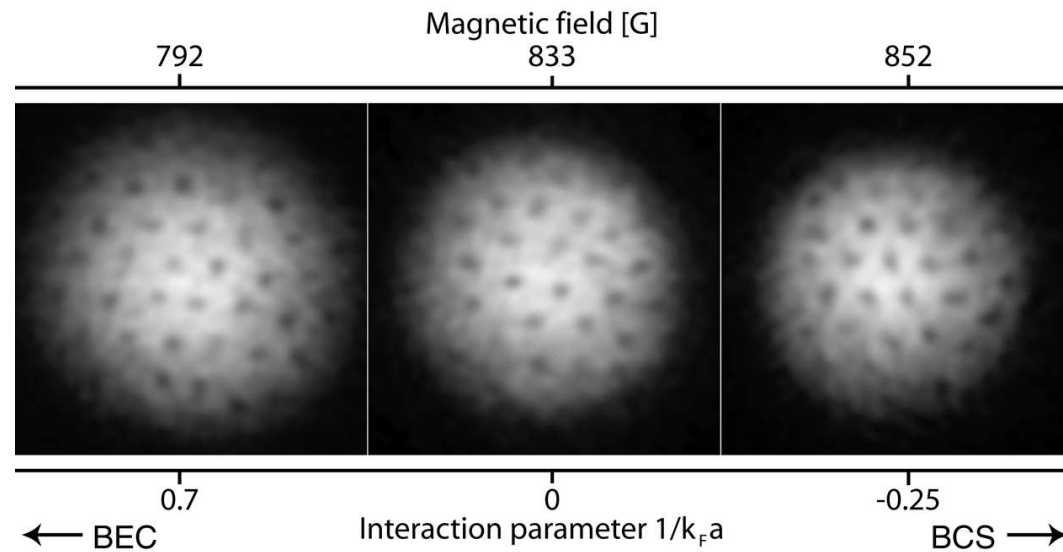


Experiments

A cloud of gas is released from the trap:



Vortices indicating superfluidity



Ground state energy

George Bertsch asked the question:

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The same parametric dependence as the energy of a free gas

$$\epsilon(n) = \xi \epsilon_{\text{free}}(n)$$

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Since then, ξ is called the Bertsch parameter.

Analogy with AdS/CFT

$$\epsilon(n) = \xi \epsilon_{\text{free}}(n)$$

Similar to pressure in $\mathcal{N} = 4$ SYM theory

$$P(T)|_{\lambda \rightarrow \infty} = \frac{3}{4} P(T)|_{\lambda \rightarrow 0}$$

Is there an useful AdS/CFT-type duality for unitarity Fermi gas?

As in $\mathcal{N} = 4$ SYM, perhaps we should start with the vacuum.

AdS/CFT correspondence

$\mathcal{N} = 4$ super-Yang-Mills theory \Leftrightarrow type IIB string theory on $\text{AdS}_5 \times \text{S}^5$.

First evidence: matching of symmetries $\text{SO}(4, 2) \times \text{SO}(6)$.

- $\text{SO}(4, 2)$: conformal symmetry of 4-dim theories (CFT_4), isometry of AdS_5
- $\text{SO}(6)$: $\sim \text{SU}(4)$ is the R-symmetry of $\mathcal{N} = 4$ SYM, isometry of S^5 .

Conformal algebra: $P^\mu, M^{\mu\nu}, K^\mu, D$

$$[D, P^\mu] = -iP^\mu, \quad [D, K^\mu] = iK^\mu$$

$$[P^\mu, K^\nu] = -2i(g^{\mu\nu} D + M^{\mu\nu})$$

Schrödinger algebra

Nonrelativistic field theory is invariant under:

- Phase rotation M : $\psi \rightarrow \psi e^{i\alpha}$
- Time and space translations, H and P^i
- Rotations M^{ij}
- Galilean boosts K^i
- Dilatation D : $\mathbf{x} \rightarrow \lambda \mathbf{x}$, $t \rightarrow \lambda^2 t$
- Conformal transformation C :

$$\mathbf{x} \rightarrow \frac{\mathbf{x}}{1 + \lambda t}, \quad t \rightarrow \frac{t}{1 + \lambda t}$$

$$[D, P^i] = -iP^i, \quad [D, K^i] = iK^i, \quad [P^i, K^j] = -\delta^{ij} M$$

$$[D, H] = -2iH, \quad [D, C] = 2iC, \quad [H, C] = iD$$

D, H, C form a $\text{SO}(2,1)$

Chemical potential $\mu\psi^\dagger\psi$ breaks the symmetry.

Generators

$$M = \int d\mathbf{x} n(\mathbf{x}), \quad P_i = \int d\mathbf{x} j_i(\mathbf{x})$$

$$K_i = \int d\mathbf{x} x_i n(\mathbf{x}), \quad C = \frac{1}{2} \int d\mathbf{x} x^2 n(x), \quad D = - \int d\mathbf{x} x_i j_i(\mathbf{x})$$

$H \rightarrow H + \omega^2 C$: putting the system in an external potential $V(\mathbf{x}) = \frac{1}{2}\omega^2 x^2$.

Operator-state correspondence

Nishida, DTS

Operator \mathcal{O} has dimension Δ if $[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0)$

$[D, K_i] = iK_i$: K_i lowers dimension by 1

$[D, C] = 2iC$: C lowers dimension by 2

Define primary operators: $[K_i, \mathcal{O}(0)] = [C, \mathcal{O}(0)] = 0$

Primary operator with dimension $\Delta \Rightarrow$ eigenstate in harmonic potential with energy $\Delta \times \hbar\omega$

Proof: let $\omega = 1$, $H_{\text{osc}} = H + C$

Define $|\Psi_{\mathcal{O}}\rangle = e^{-H}\mathcal{O}^\dagger|0\rangle$

From Schrödinger algebra one finds $e^H H_{\text{osc}} e^{-H} = C + iD$, so

$$H_{\text{osc}}|\Psi_{\mathcal{O}}\rangle = \Delta_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle$$

Cf. radial quantization of relativistic CFTs

Example

Two particles in harmonic potential: ground state with unitarity boundary condition can be found exactly

$$\Psi(\mathbf{x}, \mathbf{y}) = \frac{e^{-|\mathbf{x}-\mathbf{y}|^2}}{|\mathbf{x} - \mathbf{y}|}$$

Energy = $2\hbar\omega$.

Dimension of operator $O_2 = \psi_\uparrow\psi_\downarrow$ is 2 (naively 3)

Recall that two-particle wavefunctions behave as

$$\Psi(\mathbf{x}, \mathbf{y}) \sim \frac{1}{|\mathbf{x} - \mathbf{y}|}$$

as $\mathbf{x} \rightarrow \mathbf{y}$, the operator O_2 has to be regularized as

$$O_2 = \lim_{\mathbf{x} \rightarrow \mathbf{y}} |\mathbf{x} - \mathbf{y}| \psi_\uparrow(\mathbf{x}) \psi_\downarrow(\mathbf{y})$$

so that $\langle 0 | O_2 | \Psi(\mathbf{x}, \mathbf{y}) \rangle$ is finite.

Embedding the Schrödinger algebra

- $Sch(d)$: is the symmetry of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} + \frac{\nabla^2}{2m}\psi = 0$$

- CFT_{d+2} : is the symmetry of the Klein-Gordon equation

$$\partial_\mu^2\phi = 0, \quad \mu = 0, 1, \dots, d+1$$

In light-cone coordinates $x^\pm = x^0 \pm x^{d+1}$ the Klein-Gordon equation becomes

$$-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}\phi + \partial_i\partial_i\phi = 0, \quad i = 1, \dots, d$$

Restricting ϕ to $\phi = e^{-imx^-}\psi(x^+, \mathbf{x})$: Klein-Gordon eq. \Rightarrow Schrödinger eq.:

$$2im\frac{\partial}{\partial x^+}\psi + \nabla^2\psi = 0, \quad \nabla^2 = \sum_{i=1}^d \partial_i^2$$

This means $Sch(d) \subset CFT_{d+2}$

Embedding (2)

$Sch(d)$ is the subgroup of CFT_{d+2} containing group elements which does not change the ansatz

$$\phi = e^{imx^-} \psi(x^+, x^i)$$

Algebra: $sch(d)$ is the subalgebra of the conformal algebra, containing elements that commute with P^+

$$[P^+, O] = 0$$

One can identify the Schrödinger generators:

$$M = P^+, \quad H = P^-, \quad K^i = M^{i-},$$

$$D_{\text{nonrel}} = D_{\text{rel}} + M^{+-}, \quad C = \frac{1}{2}K^+$$

Nonrelativistic dilatation

$$\begin{array}{rcccl} & & D_{\text{rel}} & & M^{+-} \\ x^+ & \rightarrow & \lambda x^+ & \rightarrow & \lambda^2 x^+ \\ x^- & \rightarrow & \lambda x^- & \rightarrow & x^- \\ x^i & \rightarrow & \lambda x^i & \rightarrow & \lambda x^i \end{array}$$

Geometric realization of Schrödinger algebra

Start from AdS_{d+3} space:

$$ds^2 = \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2}$$

Invariant under the whole conformal group, in particular with respect to relativistic scaling

$$x^\mu \rightarrow \lambda x^\mu, \quad z \rightarrow \lambda z$$

and boost along the x^{d+1} direction:

$$x^+ \rightarrow \tilde{\lambda} x^+, \quad x^- \rightarrow \tilde{\lambda}^{-1} x^-$$

Break the symmetry down to $\text{Sch}(d)$:

$$ds^2 = \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

The additional term is invariant only under a combination of relativistic dilation and boost:

$$x^+ \rightarrow \lambda^2 x^+, \quad x^- \rightarrow x^-, \quad x^i \rightarrow \lambda x^i$$

Model

$$ds^2 = \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

Is there a model where this is a solution to the Einstein equation?

The additional term gives rise to a change in $R_{++} \sim z^{-4}$: we need matter that provides $T_{++} \sim z^{-4}$.

Can be provided by A_μ with $A_+ \sim 1/z^2$: has to be a massive gauge field.

$$S = \int d^{d+3}x \sqrt{-g} \left(\frac{1}{2}R - \Lambda - \frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu^2 \right)$$

Can be realized in string theory ($d = 2$)

Herzog, Rangamani, Ross;

Maldacena, Martelli, Tachikawa;

Adam, Balasubramanian, McGreevy (2008)

Black-hole solutions also constructed: allow studying hydrodynamics

Nonrelativistic hydro from relativistic hydro

Relativistic hydrodynamic equations

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \text{viscous term}$$

Look at solutions that are independent of x^- :

$$\begin{aligned} \partial_+ T^{++} + \partial_i T^{+i} &= 0 & \partial_t \rho + \partial_i(\rho v^i) &= 0 \\ \partial_+ T^{+i} + \partial_j T^{ij} &= 0 & \partial_t(\rho v^i) + \partial_j \Pi^{ij} &= 0 \\ \partial_+ T^{+-} + \partial_j T^{j-} &= 0 & \partial_t \epsilon + \partial_j j_\epsilon^i &= 0 \end{aligned}$$

Identify pressure on two sides: $P_{\text{rel}} = P_{\text{nonrel}}$

Identify u^+ with the density:

$$\begin{aligned} u^+ &= \sqrt{\frac{\rho}{(d+2)P}} \\ u^+ u^i &= v^i + \text{viscous term} \end{aligned}$$

Transport coefficients

Relativistic hydrodynamics: one transport coefficient η

Nonrelativistic hydrodynamics: viscosity η and thermal conductivity κ

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \kappa = 2\eta \frac{\varepsilon + P}{2\rho T}$$

“Prandtl number” equal to 1 (air: 0.7)

Rangamani, Ross, DTS, Thompson

Comparison with unitarity fermions

- $d = 2$ instead of $d = 3$; can build a model at any d [Kovtun, Nickel](#)
 - Metric is AdS_{d+3} corrected by a z^{-4} term
- No superfluid phase: needs a scalar ϕ (cf. [Herzog, Hartnoll, Horowitz; Gubser](#))
- That scalar should be dual to the operator $\psi_{\uparrow}\psi_{\downarrow}$, which has dimension 2
- Mass of ϕ can be found from dimension of ϕ
- The same mass also corresponds to another theory where $\psi_{\uparrow}\psi_{\downarrow}$ has dimension 3

$$\phi \sim Az^{\Delta} + Bz^{5-\Delta}$$

- One bulk theory corresponds to
 - $\Delta = 3$: unitarity fermions
 - $\Delta = 2$: free fermions

Conclusion

- Unitarity fermions have Schrödinger symmetry, a nonrelativistic conformal symmetry
- Universal properties, studied in experiments
- There is a metric with Schrödinger symmetry
- Starting point for dual phenomenology of unitarity fermions?
- Open questions
 - Inclusion of condensate
 - Superfluid hydrodynamics (ζ_3)
 - Rotating trap (magnetic field)