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Open/Closed Duality in Topological Matrix Models

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with Davide Gaiotto

hep-th/0312196 + work in progress

Open/Closed Duality

- Gauge theory/closed strings correspondence

AdS/CFT and related examples

Ordinary Yang-Mills?

- Large N transitions in topological string theory

Chern-Simons/A-model on the conifold Witten, Gopakumar Vafa

B-model Dijkgraaf Vafa, Aganagic Dijkgraaf Klemm Marino Vafa

- Unstable brane decay: Sen

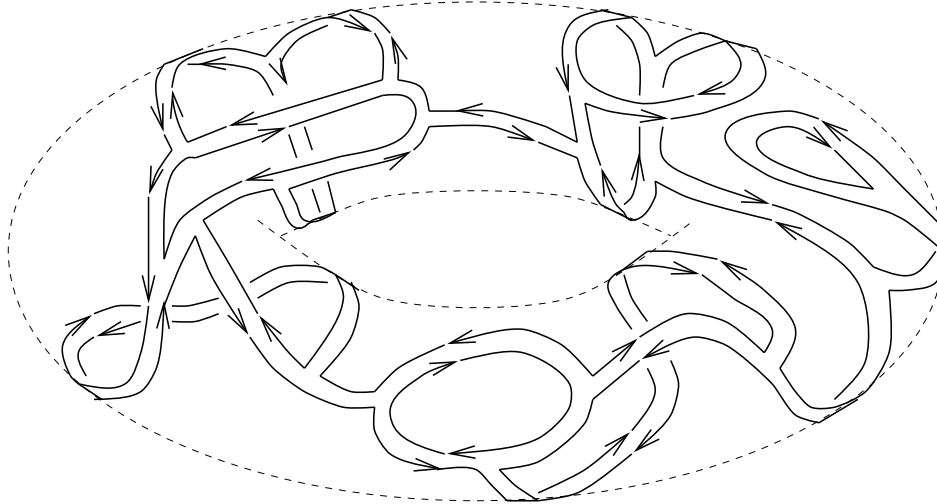
either outgoing closed radiation *or* worldvolume dynamics on the brane

Closed strings in $D = 2$ from double-scaling limit of open SFT

on $N \rightarrow \infty$ unstable branes. McGreevy Verlinde, Klebanov Maldacena Seiberg

Gauge theory partition function: 't Hooft

$$\log \mathcal{Z}^{open}(g_{YM}, N) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} (g_{YM}^2)^{2g-2} (g_{YM}^2 N)^h F_{g,h}$$



$$g_s = g_{YM}^2, \quad t = g_{YM}^2 N$$

Closed string partition function:

$$\log \mathcal{Z}^{closed}(g_s, t) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) \quad F_g(t) = \sum_{h=2}^{\infty} F_{g,h} t^h .$$

In many examples, t is a geometric parameter of the closed string background.

Riemann surfaces with h holes \leftrightarrow

Closed Riemann surfaces with h extra closed string insertions ?

D-branes in imaginary time offer a precise example Gaiotto Itzhaki L.R.

$$t b_0 \int d\rho \rho^{L_0} |\mathcal{B}\rangle_P \leftrightarrow t \mathcal{W}(P)$$

Summing over holes $\sim \exp(t \int d^2 z \mathcal{W}(z))$

Topological strings seem to work similarly Ooguri Vafa

Brief review of $c < 1$ closed string models

p	\vdots	\vdots	\vdots	\vdots	\vdots	
4	○	●	●	●	●	...
3	○	●	●	●	●	...
2	○	●	●	●	●	...
	1	2	3	4	5	q

Noncritical bosonic string theories from (p, q) minimal models + Liouville.

Exact solution from double-scaling of $p - 1$ matrix model.

Douglas Shenker, Brezin Kasakov, Gross Migdal

Models in the same row related by turning on deformations, $S = S_0 + t_n \mathcal{O}_n$.

$(p, 1)$ column: topological models

- Focus first on $(2, 1)$: $c = -2$ matter \oplus $c = 28$ Liouville.

Alternative powerful formulation as topological 2d gravity, intersection theory on moduli space of Riemann surfaces. Witten

(2, 1) model

Closed string observables \mathcal{O}_{2k+1} , $k = 0, 1, \dots, \infty$

($\mathcal{O}_{2k+1} \leftrightarrow c_1(\mathcal{L})^k$ in intersection theory)

$$\log \mathcal{Z}^{\text{closed}}(g_s, t_k) = \sum_{g=0}^{\infty} g_s^{2g-2} \langle \exp(\sum_k t_k \mathcal{O}_{2k+1}) \rangle_g .$$

$\mathcal{Z}(t) = \tau(t)$ is a τ -function of the KP(KdV) hierarchy. Douglas

Uniquely determined by Virasoro algebra of constraints: Dijkgraaf Verlinde Verlinde

$$\frac{\partial}{\partial t_k} \mathcal{Z}(t_k) = \mathcal{L}_{2k-2} \mathcal{Z}(t_k)$$

Kontsevich matrix integral

Beautiful representation of $\mathcal{Z}(t)$ from an integral over $N \times N$ hermitian matrices

$$\mathcal{Z}^{closed}(t) = \rho(\mathbf{Z})^{-1} \int [dX] \exp \left(-\frac{1}{g_o^2} \text{Tr} \left[\frac{1}{2} \mathbf{Z} X^2 + \frac{1}{6} X^3 \right] \right)$$

$$\rho(\mathbf{Z}) = \int [dX] \exp \left(-\frac{1}{2g_o^2} \text{Tr} \mathbf{Z} X^2 \right) .$$

Closed string sources t_k encoded in the N eigenvalues of matrix \mathbf{Z} :

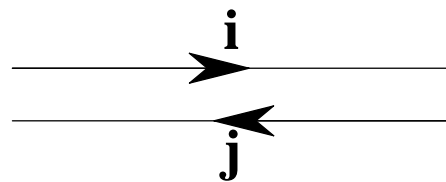
$$t_k = \frac{g_o^2}{2k+1} \text{Tr} \mathbf{Z}^{-2k-1} = \frac{g_o^2}{2k+1} \sum_{n=1}^N \frac{1}{z_n^{2k+1}}$$

Different from double-scaled matrix model.

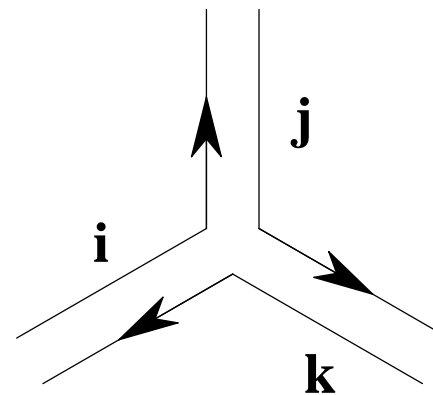
Here we can keep g_s and N finite, or do ordinary 't Hooft expansion with t_k fixed.

- Kontsevich integral resembles cubic open SFT...

Feynman rules



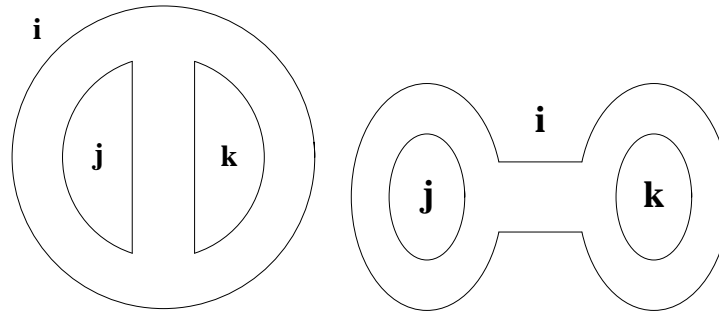
$$\frac{2g_s}{z_i + z_j}$$



$$\frac{1}{g_s}$$

Example: $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle_{S_2}$

Sphere with three holes: Assign Chan-Paton indexes to each hole i, j, k



Four contributions:

$$\frac{2g_o^2}{(z_i + z_j)(z_j + z_k)(z_k + z_i)} + \left[\frac{g_o^2}{z_i(z_i + z_j)(z_i + z_k)} + (i \rightarrow j) + (i \rightarrow k) \right] = \frac{g_o^2}{z_i z_j z_k}$$

Sum over Chan-Paton:

$$\frac{g_o^2}{6} \left(\sum_{i=1}^N \frac{1}{z_i} \right)^3 \equiv \frac{1}{6g_s^2} t_0^3 \longrightarrow \langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle_{S_2} = 1.$$

Hole in Feymann diagram \leftrightarrow closed string puncture $\sim g_s \sum \frac{\mathcal{O}_{2k+1}}{z^{2k+1}}$

- Summing over number of holes exponentiates the puncture to a closed background

Our Proposal

Kontsevich integral = open SFT on N stable D-branes

- $|\mathcal{B}\rangle_z = (\text{FZZT brane for Liouville with } \mu = 0, \mu_B = z) \otimes |\mathcal{B}^{matter}\rangle$

Introducing a brane is equivalent to a shift of the closed string background:

$$|\mathcal{B}\rangle_z \longrightarrow \sum_{k=0}^{\infty} \frac{\mathcal{O}_{2k+1}}{(2k+1)z^{2k+1}}$$

- Cubic OSFT on N such branes \rightarrow Kontsevich integral

Topological localization similar to open topological A model \rightarrow Chern-Simons Witten

- Compute \mathcal{Z} in open and closed channel:

$$\mathcal{Z}^{\text{open}}(g_o, z_i) = \mathcal{Z}^{\text{closed}} \left(g_s = g_o^2, t_{2k+1} = g_s \sum \frac{1}{(2k+1)z_i^k} \right)$$

Worldsheet theory for $(2, 1)$ model

Many formulations of topological gravity.

Careful BRST analysis necessary to define cohomological problem and handle correctly the contact term algebra

Labastida Pernici Witten, E. and H. Verlinde, Distler Nelson

Double scaling limit of a matrix model with two Grassmann coordinates agrees with topological gravity Klebanov Wilkinson

- Simplest continuum formulation: bosonic string
 $c = -2$ matter \oplus $c = 28$ Liouville

No subtleties arise for the open theory.

Precise treatment made possible by recent progress in Liouville field theory Teschner,
Fateev Zamolodchikov², ...

Strings in D=-2

$$\mathcal{S} = \frac{1}{2\pi} \int d^2z \epsilon_{\alpha\beta} \partial\Theta^\alpha \bar{\partial}\Theta^\beta + S_\phi^{c=28} + S_{bc} \quad \alpha, \beta = 1, 2.$$

Θ^1 and Θ^2 real and Grassmann odd.

$$\Theta^1(z, \bar{z})\Theta^2(0) \sim -\frac{1}{2} \log |z|^2$$

Only one *non-chiral* zero mode. Different from $\xi\eta$,

$$\eta(z) = \partial\Theta^1(z, \bar{z}), \quad \xi(z) + \bar{\xi}(\bar{z}) = \Theta^1(z, \bar{z}).$$

Closed string observables

$$\mathcal{O}_{2k+1} = e^{\sqrt{2}(1-k)\phi} \mathcal{P}_k(\partial\Theta^\alpha) c\bar{c}$$

Canonical choice of $(\frac{k(k+1)}{2}, \frac{k(k+1)}{2})$ primaries \mathcal{P}_k from SL(2) invariance.

Already in the correct “picture”.

Logarithmic behavior of the CFT possible way to understand contact terms.

(See [Zamolodchikov](#))

Stable D-branes

Boundary conditions:

- Dirichlet b.c. on Θ^α .
- Extended D-brane in the Liouville direction with boundary interaction $\mu_B \int_{\partial} e^\phi$

Fateev Zamolodchikov²

Quantum gravity interpretation

$$|\mathcal{B}\rangle_{\mu_B} \leftrightarrow \int_0^\infty e^{-\mu_B l} W(l) \sim \sum_{k=0}^{\infty} \frac{\mathcal{O}_{2k+1}}{\mu_B^{2k+1}}$$

$W(l)$ macroscopic loop operator Martinec Moore Seiberg Staudacher

- \mathcal{O}_{2k+1} appear with correct power of μ_B in Boundary State
if $\mu_B \leftrightarrow z$ parameter in Kontsevich.

BCFT on stable branes

Spectrum on these branes:

- Θ^α : Dirichlet b.c. \rightarrow a **single copy** of current $\partial\Theta^\alpha$, no zero modes
- Boundary (FZZT) Liouville: $\{e^{\alpha\phi}\}$

$\alpha = Q/2 + iP$ are normalizable states;

α real $\leq Q/2$ are local operators. $(c_{Liou} \equiv 1 + 6Q^2, Q = b + 1/b, \quad b = 1/\sqrt{2})$

Bosonization **Distler**

$$\beta = \partial\Theta^1 e^{b\phi} \quad \gamma = \partial\Theta^2 e^{-b\phi}$$

$(2,-1)$ $\beta\gamma$ system, $(2,-1)$ bc system

Scalar supercharge

$$Q_S = \oint b(z)\gamma(z) = \oint b(z)e^{-\phi(z)}\partial\Theta^2(z),$$

$$Q_S^2 = \{Q_B, Q_S\} = \{b_0, Q_S\} = [L_n, Q_S] = 0.$$

$$L_0 = \{Q_S, \dots\}$$

Open String Field Theory

Usual OSFT action on N D-branes: Witten

$$S[\Psi] = -\frac{1}{g_o^2} \left(\frac{1}{2} \langle \Psi_{ij}, Q_B \Psi_{ji} \rangle + \frac{1}{3} \langle \Psi_{ij}, \Psi_{jk}, \Psi_{ki} \rangle \right) .$$

String field $\Psi_{ij} \in \mathcal{H}_{ij}$, open string state-space between brane i and brane j .

Topological localization

- $\mathcal{S}_{WS} = \{Q_S, \dots\} \rightarrow$ localization on ‘massless’ modes
- More formal argument:

Gauge-fix action with Siegel gauge $b_0\Psi_{ij} = 0$. Extend Ψ_{ij} to arbitrary ghost number

Q_S cohomology is one-dimensional: open string tachyon $c e^{b\phi}$.

OSFT action Q_S closed.

$$\Psi_{ij} = X_{ij}T_{ij} + \dots = X_{ij}e^{b\phi}c_1|0\rangle_{ij} + \dots$$

$$S[\Psi] = -\frac{1}{g_o^2} \left(\frac{1}{2} X_{ij} X_{ij} \langle T_{ij}, c_0 L_0 T_{ji} \rangle + \frac{1}{3} X_{ij} X_{jk} X_{ki} \langle T_{ij}, T_{jk}, T_{ki} \rangle \right) + Q_S(\dots)$$

Vacuum amplitudes containing states outside the cohomology add up to zero.

Boundary Liouville correlators

Naive computation: Liouville momentum has to add up to 3.

$$\langle T_{ij}, c_0 L_0 T_{ij} \rangle = \mu_B^{(i)} + \mu_B^{(j)}. \quad (\text{Needs one } \mu_B e^\phi \text{ insertion})$$

$$\langle T_{ji}, T_{jk} * T_{ki} \rangle = 1 \quad (\text{Needs no } \mu_B \text{ insertion})$$

Kontsevich model: $\mu_B = z$

Kinetic term would be naively zero (open tachyon is on-shell). Why non-zero?

Compute carefully with FZZT formulae:

$$\langle e^{\alpha\phi} e^{\alpha\phi} \rangle_{1,2} = D(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu_{bulk})$$

$D(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu_{bulk})$ has a pole as $\alpha \rightarrow 1$ that cancels against $L_0 \rightarrow 0$.

All contributions come from singular surfaces

In cell decomposition of OSFT, all propagator lengths $L_i \rightarrow \infty$

Evaluating Boundary Liouville correlators

$$D(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu_{bulk}) = \left(\frac{\pi}{\sqrt{2}} \mu_{bulk} \gamma\left(\frac{1}{2}\right) \right)^{\frac{1}{2}-\alpha} \times$$

$$\times \frac{\Gamma_{\frac{1}{\sqrt{2}}}(\sqrt{2}\alpha - \frac{3}{\sqrt{2}}) S_{\frac{1}{\sqrt{2}}}(\frac{3}{\sqrt{2}} + i s_1 + i s_2 - \sqrt{2}\alpha) S_{\frac{1}{\sqrt{2}}}(\frac{3}{\sqrt{2}} - i s_1 - i s_2 - \sqrt{2}\alpha)}{\Gamma_{\frac{1}{\sqrt{2}}}(\frac{3}{\sqrt{2}} - \sqrt{2}\alpha) S_{\frac{1}{\sqrt{2}}}(i s_1 - i s_2 + \sqrt{2}\alpha) S_{\frac{1}{\sqrt{2}}}(-i s_1 + i s_2 + \sqrt{2}\alpha)}$$

$$\frac{\mu_B^{(i)}}{\sqrt{\mu_{bulk}}} = \cosh \sqrt{2}\pi s_i$$

Contact terms

Open string contact terms: boundaries touch each other

Closed string contact terms: punctures touch each other

Highly nontrivial contact terms in closed string description give rise to recursion relations for amplitudes.

$$\frac{\partial}{\partial t_1} \mathcal{Z} = \mathcal{L}_{-2} \mathcal{Z} \equiv \frac{t_1^2}{2g_s^2} \mathcal{Z} + \sum_{k=0}^{\infty} (2k+3)t_{2k+3} \frac{\partial \mathcal{Z}}{\partial t_{2k+1}}$$

$$\frac{\partial}{\partial t_3} \mathcal{Z} = \mathcal{L}_0 \mathcal{Z} \equiv \frac{1}{8} \mathcal{Z} + \sum_{k=0}^{\infty} (2k+1)t_{2k+1} \frac{\partial \mathcal{Z}}{\partial t_{2k+1}}$$

$$\frac{\partial}{\partial t_{2n+5}} \mathcal{Z} = \mathcal{L}_{2n+2} \mathcal{Z} \equiv \sum_{k=0}^{\infty} (2k+1)t_{2k+1} \frac{\partial \mathcal{Z}}{\partial t_{2k+2n+1}} + \frac{g_s^2}{2} \sum_{k=0}^n \frac{\partial^2 \mathcal{Z}}{\partial t_{2k+1} \partial t_{2n-2k+1}}$$

Recursion encode contact terms for \mathcal{O}_{2n+1} at position of other \mathcal{O}_{2m+1} and nodes of surface. Witten, Dijkgraaf Verlinde Verlinde

Non-trivial self-consistency: Virasoro algebra.

Extended recursion relations

How do boundaries affect recursion relations?

New contact terms for \mathcal{O}_{2n+1}

at boundaries and at nodes that pinch a boundary.

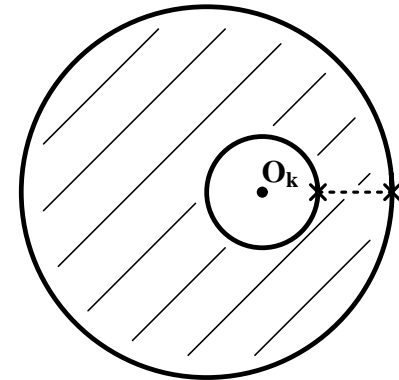
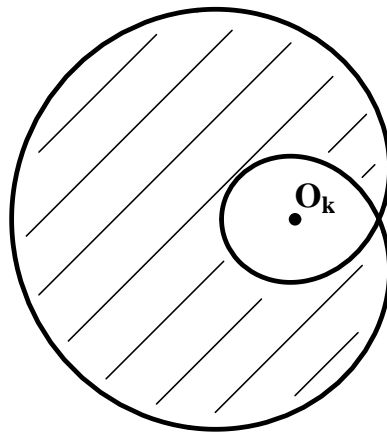
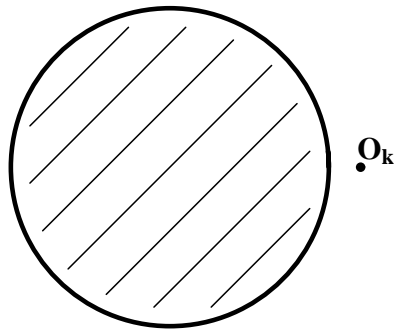
$$\frac{\partial}{\partial t_1} \mathcal{Z} = \tilde{\mathcal{L}}_{-2}^{(z)} \mathcal{Z} \equiv \mathcal{L}_{-2} \mathcal{Z} + \left(\frac{t_1}{z g_s} + \frac{1}{2z^2} \right) \mathcal{Z} - \frac{1}{z} \frac{\partial \mathcal{Z}}{\partial z}$$

$$\frac{\partial}{\partial t_3} \mathcal{Z} = \tilde{\mathcal{L}}_0^{(z)} \mathcal{Z} \equiv \mathcal{L}_0 \mathcal{Z} - z \frac{\partial \mathcal{Z}}{\partial z}$$

$$\frac{\partial}{\partial t_{2n+5}} \mathcal{Z} = \tilde{\mathcal{L}}_{2n+2}^{(z)} \mathcal{Z} \equiv \mathcal{L}_{2n+2} \mathcal{Z} - z^{2n+1} \frac{\partial \mathcal{Z}}{\partial z} - g_s \sum_{k=0}^n z^{2k+1} \frac{\partial \mathcal{Z}}{\partial t_{2n-2k+1}}$$

Unique solution:

$$\mathcal{Z}_{open+closed}(t_{2k+1}, z) = \mathcal{Z}_{closed} \left(t_{2k+1} + \frac{g_s}{(2k+1)z^{2k+1}} \right)$$



In this model, all the physics can be extracted from the open string vacuum amplitudes on an infinite number of branes

This may be the correct framework for background independence Witten

- Generalizations to other $c \leq 1$ and $\hat{c} \leq 1$

Non-zero bulk cosmological constant

Treat closed string deformation $\mu e^{2b\phi}$ perturbatively.

In OSFT, add closed string insertions with **open/closed vertex**. In this simple case,

$$\mathcal{Z}(g_s, \mu, z_i) = \mathcal{Z}_{closed}(g_s, \mu) \rho(Z)^{-1} \int [dX] \exp \left(\frac{1}{g_s} \text{Tr} \left[-\frac{1}{2} Z X^2 + \frac{1}{6} X^3 + \mu X \right] \right)$$

Shift $X \rightarrow X - (Z^2 - 2\mu)^{\frac{1}{2}} + Z$ gives

$$\int [dX] \exp \left(\frac{1}{g_s} \text{Tr} \left[-\frac{1}{2} (Z^2 - 2\mu)^{\frac{1}{2}} X^2 + \frac{1}{6} X^3 \right] \right)$$

Hence $\mu_B \equiv (z^2 - 2\mu)^{\frac{1}{2}}$, which checks out.

Similarly, **dilaton** deformation

$$t_3 \mathcal{O}_3 \rightarrow \text{open/closed vertex } \frac{3t_3}{g_s} \text{Tr} Z^2 X.$$

Higher \mathcal{O}_k have non-trivial contact terms \rightarrow multi-trace deformations?

Generalization to $(p, 1)$ (work in progress)

Kontsevich integral for $(p, 1)$: One matrix with action of order $p + 1$ $KMMMZ$

Topological OSFT $(p, 1)$: $p - 1$ matrices with cubic action

Example: $(3, 1)$

$$\mathcal{Z}^{KMMMZ}(t) = \rho(\mathbf{Z})^{-1} \int [dX] \exp \left(-\frac{1}{g_o^2} \text{Tr} \left[\frac{1}{4} \mathbf{Z}^2 X^2 + \frac{1}{6} \mathbf{Z} X^3 + \frac{1}{24} X^4 \right] \right)$$

$$\mathcal{Z}(t) \sim \int [dX][dY] \exp \left(-\frac{1}{g_o^2} \text{Tr} [a_1 \mathbf{Z}^2 X^2 + a_2 \mathbf{Z} X Y + a_3 Y^2 + b_1 \mathbf{Z} X^3 + b_2 X^2 Y] \right)$$

CFT: twisted $SU(2)_{p-2} \oplus \Theta_\alpha \oplus \text{Liouville} \oplus bc \cong \text{top. MM} \oplus \text{top. gravity}$

Structure of matrix model dictated by Liouville dressing

Y can be integrated away \rightarrow quartic potential: $\mathcal{Z}(t) = \mathcal{Z}^{KMMMZ}(t)$?

Conclusions

A new class of examples of exact open/closed duality

Prototype: Kontsevich model

Many possible extensions:

- (p, q) minimal models
- $c = 1$ at self-dual radius Mukhi Murthy
- $\hat{c} < 1$ models?
- critical topological strings?

General lesson:

Open SFT on infinite number of branes as a universal tool for open/closed dualities