Yang-Mills Theory in (2+1) Dimensions

Vacuum wave function, string tension, etc.

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Why is YM(2+1) interesting?

- Interesting in its own right

YM(1+1)	YM(2+1)	YM(3+1)
No propagating degrees of freedom Exactly solvable	Propagating degrees of freedom, Nontrivial Dimensional coupling Super-renormalizable	Highly nontrivial Too difficult

- A real physical context for YM(2+1)

Mass gap of YM(2+1) $\approx$ Magnetic screening mass of YM(3+1) at high temperatures

Magnetic screening necessary to define thermal perturbation theory due to magnetic-type infrared divergences

$$\text{YM}(3+1) \longrightarrow \quad \text{YM}(3) \quad \text{as } T \to \infty$$

$$\longrightarrow \quad \text{YM}(2+1) \text{ by Wick rotation}$$

We can get an estimate of magnetic mass

Work in collaboration with D. Karabali and Chanju Kim

We use a Hamiltonian approach, some exact results are possible.

Hamiltonian Analysis

Choose $A_0 = 0 \longrightarrow \ A_i,\ i = 1, 2.$

$$A_i^g = g\ A_i\ g^{-1}\ -\ \partial_i g\ g^{-1}$$

Wave functions are gauge-invariant (Equivalent to imposing Gauss law)

Requires 3 basic ingredients

1. Inner product
   - Matrix variables
   - Gauge-invariant measure
   - Identification of proper gauge-invariant variables (CFT argument)

2. Hamiltonian $\mathcal{H}$ in the new variables
   (Propagator mass, comparison with resummation techniques,...)

3. Solve the Schrödinger equation $\mathcal{H}\Psi = E\Psi$

$\longrightarrow$ Ψ_0, the vacuum wave function

$\longrightarrow$ string tension, comparison with lattice estimates

Matrix variables, volume element

Complex coordinates, $z = x_1 - ix_2, \quad \bar{z} = x_1 + ix_2$

$$A \equiv A_z = \frac{1}{2}(A_1 + iA_2), \qquad \bar{A} = \frac{1}{2}(A_1 - iA_2)$$

1. Parametrize A as

$$A = -\partial M \; M^{-1} \qquad \bar{A} = M^{\dagger -1} \bar{\partial} M^{\dagger}$$

$$G = SU(N) \implies \quad M \in SL(N, \mathbf{C}) = SU(N)^{\mathbf{C}}$$

More generally $G \to G^{\mathbf{C}}$

Parametrization well-known in 2 dim. YM context, gauged WZW models, etc.

M and $MV(\bar{z}) \implies$ same A (Come back to this later)

2. Gauge transformation

$$M \longrightarrow M^g = g\ M, \qquad A_i^g = gA_ig^{-1} - \partial_i g g^{-1}$$

$H = M^\dagger M$ is gauge-invariant

3. Calculation of volume

$$\begin{aligned} \delta A &= -\partial(\delta M M^{-1}) + [\partial M M^{-1}, \delta M M^{-1}] \\ &= -D(\delta M M^{-1} \\ \delta \bar{A} &= \bar{D}(M^{\dagger -1} \delta M^\dagger) \end{aligned}$$

$$\begin{aligned} ds^2_{\mathcal{A}} &= \int d^2x \ \mathrm{Tr}(\delta A \delta \bar{A}) \\ &= \int \mathrm{Tr}\left[(M^{\dagger -1}\delta M^\dagger)(-\bar{D}D)(\delta M M^{-1})\right] \\ ds^2_{SL(N,\mathbf{C})} &= \int \mathrm{Tr}(M^{\dagger -1}\delta M^\dagger \ \delta M M^{-1}) \end{aligned}$$

$$d\mu_{\mathcal{A}} = \det(-\bar{D}D) \underbrace{d\mu(M, M^{\dagger})}_{\text{Haar measure for } SL(N,\mathbf{C})}$$

$$d\mu(M, M^{\dagger}) = \underbrace{d\mu(U)}_{\text{Haar for } SU(N)} \underbrace{d\mu(H)}_{\text{Haar for } \frac{SL(N,\mathbf{C})}{SU(N)}}$$

$$H = M^{\dagger}M, \quad H = e^{t^a \varphi^a}, \quad H^{-1}\delta H = \delta\varphi^a R_{ab}(\varphi) t^b$$

$$d\mu(H) = [d\varphi] \det R$$

$$d\mu_{\mathcal{A}} = \det(-\bar{D}D)\, d\mu(H)\, d\mu(U)$$

$$d\mu(\mathcal{A}/\mathcal{G}) = \det(-\bar{D}D)\, d\mu(H)$$

$$= d\mu(H)\, \exp\left[2C_A\, S_{wzw}(H)\right]$$

$C_A \delta_{ab} = f_{amn} f_{bmn} = N\delta_{ab}$ for $SU(N)$.

$$S_{wzw}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) - \frac{i}{12\pi}\int \mathrm{Tr}(H^{-1}dH)^3$$

(WZW action for H)

4. Inner product

$$\langle 1|2\rangle = \int d\mu(H)\ \exp\left[2C_A\ S_{wzw}(H)\right]\quad \Psi_1^* \Psi_2$$

<u>*Two remarks*</u>

1. YM (2+1) has Gribov problem. But inner product formula has no difficulty due to this, it is exact
2. It shows

Matrix elements in YM(2+1) = Correlators of a hermitian WZW model

An intuitive argument for mass gap

$$\mathcal{H} = \frac{1}{2}\int \left[e^2 E^2 + \frac{B^2}{e^2}\right]$$

$$[E, B] \sim p \qquad \text{(in momentum space)}$$

$$\Delta E \; \Delta B \sim p, \qquad \Delta E \sim \frac{p}{\Delta B}$$

$$\mathcal{E} \approx \frac{1}{2}\left[e^2 \frac{p^2}{(\Delta B)^2} + \frac{(\Delta B)^2}{e^2}\right]$$

Minimize with respect to $\Delta B \; \rightarrow (\Delta B)^2 \sim p$

$\rightarrow \; \mathcal{E} \sim p$. This is the photon.

For us,

$$\langle \mathcal{H} \rangle = \int d\mu(H)\, \exp\left[2C_A S_{wzw}(H)\right] \frac{1}{2} \int \left[e^2 E^2 + \frac{B^2}{e^2}\right]$$

$$\approx \int d\mu(H)\, \exp\left[-\frac{C_A}{2\pi} \int B \frac{1}{p^2} B + \ldots\right] \frac{1}{2} \int \left[e^2 E^2 + \frac{B^2}{e^2}\right]$$

Gaussian $\longrightarrow$ $(\Delta B)^2 \sim p^2$ $\longrightarrow$ mass gap.

More detailed analysis $\longrightarrow$

$$m_{mag} = m = \frac{g^2 T C_A}{2\pi}$$

Calculation of $\det(-\bar{D}D)$

$$\Gamma = \log\det(-\bar{D}D) = \mathrm{Tr}\log(-\bar{D}D)$$

$$\frac{\delta\Gamma}{\delta A^a(x)} = \mathrm{Tr}\left[D^{-1}_{Reg}(x,y)(-it^a)\right]_{y\to x}$$

$$= \mathrm{Tr}\,\left[\mathcal{G}(x,x)(-it^a)\right]$$

$$= \frac{1}{\pi}\,\mathrm{Tr}\,[\bar{\partial}M M^{-1}(-it^a)]$$

Re-integrate to get the determinant.

The hermitian WZW model

Unitary model	Hermitian model
Level k	Level $k + 2C_A$
$\exp[kS_{wzw}(U)]$	$\exp[(k + 2C_A)S_{wzw}(H)]$
$\kappa = k + C_A$	$\kappa = -(k + C_A)$
	Compare using $(\kappa \leftrightarrow -\kappa)$
Integrable rep's	
$\sim$ spin $\leq k$	
Nonintegrable	Nonintegrable
$\rightarrow$ Correlators $=0$	$\rightarrow$ Correlators $= \infty$
	"Finite norm" $\Longrightarrow$
	Integrable rep's

$k = 0$ for us, $\Longrightarrow$ Ψ's are functions of the current

$$J = \frac{C_A}{\pi} \partial H \; H^{-1}$$

$$W(C) = \mathrm{Tr}\; \mathcal{P} e^{-\oint A} = \mathrm{Tr}\; \mathcal{P} \exp\left(\frac{\pi}{C_A} \oint J \right)$$

All gauge-invariant quantities can be made from J.

Construction of $\mathcal{H}$

$$\mathcal{H} = \underbrace{\frac{e^2}{2}\int E^a E^a}_{} + \underbrace{\frac{1}{2e^2}\int B^a B^a}_{}$$

$$\equiv \quad T \quad + \quad V$$

$$T\ \Psi = -\frac{e^2}{2}\int_x \frac{\delta^2}{\delta A(x)\delta \bar{A}(x)}\ \Psi$$

$$= -\frac{e^2}{2}\left[\int \underbrace{\frac{\delta J(u)}{\delta A(x)}\ \frac{\delta J(v)}{\delta \bar{A}(x)}}_{\color{red}{\Omega}}\ \frac{\delta^2 \Psi}{\delta J(u)\delta J(v)} + \int \underbrace{\frac{\delta^2 J(u)}{\delta A(x)\delta \bar{A}(x)}}_{\color{red}{\omega}}\ \frac{\delta \Psi}{\delta J(u)}\right]$$

$$= \int \Omega_{ab}(u,v)\frac{\delta^2 \Psi}{\delta J^a(u)\delta J^b(v)} + \int \omega^a(u)\frac{\delta \Psi}{\delta J^a(u)}$$

$$\omega^a = -\frac{e^2}{2}\int_x \frac{\delta^2 J^a(u)}{\delta A^b(x)\delta \bar{A}^b(x)}$$

$$= \left(\frac{e^2 C_A}{2\pi}\right) M^\dagger_{am}(x)\mathrm{Tr}\left[t^m \bar{D}^{-1}(y,x)\right]_{y\to x} \qquad \leftarrow \text{cf. } d\mu(\mathcal{A}/\mathcal{G})$$

$$= m\ J^a \qquad\qquad m = \frac{e^2 C_A}{2\pi}$$

$$T = m\left[\int J^a \frac{\delta}{\delta J^a} + \int \Omega_{ab}(u,v)\frac{\delta^2}{\delta J^a(u)\delta J^b(v)}\right]$$

$$\Omega_{ab}(u,v) = \frac{C_A}{\pi^2}\frac{\delta_{ab}}{(u-v)^2} - i\frac{f_{abc}J^c(v)}{u-v} + \mathcal{O}(\epsilon)$$

Recheck $\int J\frac{\delta}{\delta J}$ by self-adjointness.

$$V = \frac{1}{2e^2}\int B^a B^a = \frac{2\pi^2}{e^2 C_A^2}\int :\bar\partial J^a \bar\partial J^a: \\ = \frac{\pi}{m C_A}\int :\bar\partial J\ \bar\partial J:$$

Regularization

$$\overline{\mathcal{G}}_{ma}(x,y) = \frac{1}{\pi(x-y)}\left[\delta_{ma}\ -\ e^{-\frac{(x-y)^2}{\epsilon}}\ [H(x,\bar y)H^{-1}(y,\bar y)]\right]$$

All results checked using regularization

Useful result

$$T\ V = 2m\ V$$

Vacuum wave function

Ignore V for the moment

Vacuum: $\Psi_0 = 1$, this is okay since $T\ \Psi_0 = 0$

Normalizable $\quad \int d\mu(H) e^{2C_A S_{wzw}(H)}\ \Psi_0^* \Psi_0 < \infty$

Include V perturbatively, $\quad \Psi_0 = e^P$

$$P = -\frac{\pi}{m^2 C_A} \int :\bar{\partial} J \bar{\partial} J : \\ - \left(\frac{\pi}{m^2 C_A}\right)^2 \int :\bar{\partial} J \mathcal{D} \bar{\partial}\ \bar{\partial} J : \quad + \frac{1}{3} \int :\bar{\partial} J [J, \bar{\partial}^2 J] : \\ + \cdots \qquad\qquad\qquad\qquad + \cdots$$

sum (first column of terms) $\qquad$ *sum* (second column of terms)

$$P = -\frac{2}{e^2}\left[\frac{\pi^2}{C_A^2}\int \bar\partial J^a(x)\left(\frac{1}{m+\sqrt{m^2-\nabla^2}}\right)_{x,y}\bar\partial J^a(y)\right.$$
$$\left. + f_{abc}\int J^a(x)J^b(y)J^c(z)f(x,y,z) + \mathcal{O}(J^4)\right]$$

$$f(x,y,z) = \int e^{ikx+ipy+iqz}(2\pi)^2\delta(k+p+q)$$
$$\times\left(\frac{\pi}{2C_A}\right)^3 \frac{(\sqrt{k^2+m^2}-m)(\sqrt{p^2+m^2}-m)}{\sqrt{k^2+m^2}+\sqrt{p^2+m^2}+\sqrt{q^2+m^2}}\ \frac{\bar k - \bar p}{kp}$$

$$\Psi_0 = e^P \approx \exp\left[-\frac{1}{2e^2}\int B\frac{1}{\sqrt{-\nabla^2}}B\right] \qquad \frac{k}{m} \gg 1$$
$$\approx \exp\left[-\frac{1}{4e^2 m}\int B^2\right] \qquad \frac{k}{m} \ll 1$$

$\mathcal{O}(J^3, J^4)$ terms are small at $k \gg e^2$ and at $k \ll e^2$

String tension

$$\langle W_R(C) \rangle = \langle \mathrm{Tr}_R \, \mathcal{P} \, e^{\frac{\pi}{C_A} \oint J} \rangle$$

$$\approx (\text{constant}) \; e^{-\sigma \mathcal{A}_C}$$

$$\sqrt{\sigma} = e^2 \sqrt{\frac{C_A C_R}{4\pi}}$$

C_R= Casimir for representation R
C_A= Casimir for the adjoint representation

Consistent with large N expectations, even though we did not use large N analysis

Comparison with lattice calculations

Compare $\sqrt{\sigma}/e^2$ with numerical (lattice)estimates by Teper et al. Our predictions are in black, lattice values are in red. (difference $\leq 3\%$)

Group	k=1 Fund.	k=2 antisym	k=3 antisym	k=2 sym	k=3 sym	k=3 mixed
$SU(2)$	0.345 0.335					
$SU(3)$	0.564 0.553					
$SU(4)$	0.772 0.759	0.891 0.883		1.196 1.110		
$SU(5)$	0.977 0.966					
$SU(6)$	1.180 1.167	1.493 1.484	1.583 1.569	1.784 1.727	2.318 2.251	1.985 1.921

Comment on higher corrections

Contribution from $\mathcal{O}(J^3)$, etc.?

Two types of corrections possible

- Corrections to coupling, purely numerical. $\longrightarrow$ ratios σ_R/σ_F are unaffected
- Corrections via new diagrams to Wilson line expectation value (under investigation)

Magnetic mass, resummed perturbation theory

$$T = m\left[\int J\frac{\delta}{\delta J} + \Omega\frac{\delta}{\delta J}\frac{\delta}{\delta J}\right] \implies T\ J^a = m\ J^a$$

$$(T+V)\ J^a = \sqrt{p^2+m^2}J^a\ + \cdots$$

$$\Psi \to e^{-C_A S_{wzw}(H)}\Psi, \qquad \mathcal{H} \to e^{-C_A S_{wzw}(H)}\ \mathcal{H}\ e^{-C_A S_{wzw}(H)}$$

$$\mathcal{H} = \frac{1}{2}\int\left[-\frac{\delta^2}{\delta\phi^2} + \phi(-\nabla^2+m^2)\phi + \cdots\right]$$

Propagator mass for gauge particles (magnetic mass)

$$m = \frac{e^2 C_A}{2\pi} \approx 0.32\ e^2 \qquad\qquad \text{for } SU(2)$$

Comparison with other methods

$m/e^2 =$	0.35	Common factor for glueball masses (lattice, Philipsen)
	0.51	Max. Abelian gauge (lattice, Karsch et al)
	0.52	Landau gauge (")
	0.44	$\lambda_3 = 2$ gauge (")
	0.38	Resummation of P.T. (Alexanian & Nair)
	0.28	Resummation of P.T. (Buchmuller & Philipsen, Jackiw & Pi)
	0.37	Gauge-invariant lattice lattice definition (Philipsen)

Glueballs

$$T\ J^a = m\ J^a$$

J^a is not a good state, need holomorphic invariance.

$$A = -\partial M M^{-1}, \qquad M,\ MV(\bar{z}) \longrightarrow \text{same } A$$

$$J \rightarrow VJV^{-1} - \partial V V^{-1} \qquad \bar{\partial} J \rightarrow V\ \bar{\partial} J\ V^{-1}$$

$\underline{A\ 2J - state}$ (0^{++})

$$\Psi_2 = \int f(x,y) : \bar{\partial} J^a(x)\ [H(x,\bar{y})H^{-1}(y,\bar{y})]_{ab}\ \bar{\partial} J^b(y)$$

Take the same $x, y \rightarrow f(x,y) = \sigma(x,y,\epsilon)$

$$T\ \Psi_2 = 2m\ \Psi_2$$

Higher number of J's

$$\Psi_n \ \sim \ :\bar{\partial} J^{a_1}\, \bar{\partial} J^{a_2} \cdots \bar{\partial} J^{a_n} :\ \underbrace{\omega_{a_1 a_2 \cdots a_n}}_{\text{invariant tensor of } SU(N)}$$

$$T\ \Psi_n = mn\ \Psi_n$$

Can include center-of-mass motion from $\int B^2/2e^2$.
Relative motion $\longrightarrow$ higher "radial" excitations $\longrightarrow$ Regge trajectory

Back to Ψ_2:

$$\left\{ 2m + \left(\frac{-\nabla_1^2}{2m} + \frac{-\nabla_2^2}{2m} \right) + m \log \frac{|x-y|}{2\epsilon} + \cdots \right\} f(x,y) = E\, f(x,y)$$

Yang-Mills +Chern-Simons

$$\Psi = \exp\left[k S_{wzw}(M^\dagger) - \frac{k}{4\pi}\int A^a \bar{A}^a\right]\ \Phi(H)$$

$$\langle 1|2\rangle = \int d\mu(H) e^{(k+2C_A)S_{wzw}(H)}\ \Phi_1^* \Phi_2$$

$$T = \frac{e^2}{4\pi}(k+2C_A)\int J\frac{\delta}{\delta J}\ +\ \frac{e^2 C_A}{2\pi}\int \Omega \frac{\delta}{\delta J}\frac{\delta}{\delta J}$$

$$\Phi_0 \approx \exp\left[-\frac{\pi}{mC_A}\int \bar{\partial} J \frac{1}{\tilde{m}+\sqrt{\tilde{m}^2-\nabla^2}}\bar{\partial} J\right]$$

$$\tilde{m} = \frac{e^2}{4\pi}(k+2C_A)$$

- New integrable operators from CFT $\longrightarrow$ screening of $W_F(C)$
- Large k $\longrightarrow$ standard perturbation theory
- A number of eigenstates of T can be constructed

Comment on Gribov problem

$$\begin{array}{ccc} \mathcal{G}_* \to & \mathcal{A} & \\ & \downarrow & \\ & \mathcal{A}/\mathcal{G}_* & \end{array}$$

Nontrivial bundle

$$\Pi_2(\mathcal{A}/\mathcal{G}_*) = \mathbf{Z} \qquad \Pi_n(\mathcal{A}) = 0$$

$$\longrightarrow \text{Noncontractible } S^2 \text{ in } \mathcal{A}/\mathcal{G}_*$$

$$H = \cosh 2f + \mathcal{J} \sinh 2f$$

$$f = \frac{1}{2} \log \left(\frac{z\bar{z} + w\bar{w} + \mu^2}{z\bar{z} + w\bar{w}} \right)$$

$$\mathcal{J} = \begin{pmatrix} z\bar{z} - w\bar{w} & 2\bar{w}z \\ 2w\bar{z} & w\bar{w} - z\bar{z} \end{pmatrix}$$

$w\bar{w} = 0$, $z = 0$ singular point. Move singularity by

$$H \to VH\bar{V}, \qquad \bar{V} = \exp\left[\sigma_3 \left(\log \frac{\bar{z}}{\bar{z} - \bar{a}}\right)\right]$$

$S_{wzw}(H)$ unchanged, finite

What is next?

Many things are still not understood

- Is a proof of mass gap possible?
- Better handle on glueballs
- Higher order corrections (*)
- Screening of adjoint and string breaking (*)
- YM(3+1), do we dare?