What is F-theory?

David R. Morrison

University of California, Santa Barbara

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Introduction

In this talk, I will formulate F-theory in a way which emphasizes its close connection to type IIB supergravity as well as its differences from type IIB string theory. My formulation gives an intrinsic description of F-theory and does not rely on dualities with other theories. The $SL(2, \mathbb{Z})$ invariance which is characteristic of F-theory will be implemented by using the classical theory of elliptic integrals together with some features of $SL(2, \mathbb{Z})$ which were only understood in modern times.

Although this formulation provides a much broader class of F-theory vacua than has previously been available, there are a few things which it misses that I will discuss at the end of the talk.

But first, we ask an easier question: what is string theory?

What is string theory?

A string theory is a ten-dimensional quantum theory of gravity semi-classcially approximated by:

- 10D SUGRA with quantized *p*-form fields, coupled either to 10D SYM (heterotic cases) or to gauge fields localized on branes (type I & II cases) with additional massless fields associated to brane intersections.

(Mild singularities may be allowed in the background spacetime, which may introduce additional degrees of freedom.)

Such a theory has a string coupled to the NS-NS 2-form field, whose coupling is given by one of the scalars in the theory. There are five weakly-coupled string theories.

The background fields in the theory consist of

- some *p*-form fields, as dictated by the SUGRA theory
- (nonabelian) gauge fields (in spacetime or on the branes)
- a metric (with specified asymptotics near branes)
- fermions

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Brane asymptotics

Many of these theories have Dp-branes, which are (Lorentz-signature) defects in spacetime of dimension $p+1$. Let M^{p+1} be the brane worldvolume , and let I^{9-p} denote the transverse space near the brane. The asymptotic behavior of the spacetime metric near a stack of N Dp-branes is a warped product

$$ds^2 = \varphi(y) dx_M^2 + dy_I^2$$

accompanied by a field strength for the SUGRA $(7-p)$-form field:

$$[d\varphi \wedge d\,\mathrm{vol}_M + \star\,(d\varphi \wedge d\,\mathrm{vol}_M)]_{8-p}\,.$$

(The are two terms to enforce self-duality in case $p=3$.) The equation of motion for the warping factor $\varphi(y)$ is

$$\Delta(\varphi) = \text{const } N\,\delta(y - y_0).$$

There is also a $\mathfrak{u}(N)$ gauge theory on the brane worldvolume.

Dualities in String Theory

Various "dualities" related the string theories to each other. We focus on strong-coupling behavior.

- S-duality of type IIB
  When combined with the shift symmetry of the R-R scalar field, this generates $\mathrm{SL}_2(\mathbb{Z})$
- S-duality relating type I to $\mathfrak{so}(32)$-heterotic
- The strong coupling limits of $(\mathfrak{e}_8 \oplus \mathfrak{e}_8)$-heterotic and type IIA are not string theories.

These last two strong coupling limits are usually interpreted in terms of M-theory.

What is M-theory?

M-theory is an eleven-dimensional quantum theory of gravity semi-classcially approximated by:

- 11D SUGRA with quantized *p*-form fields, coupled to 6D (2,0) theories along 5-branes, and to certain 3D theories along 2-branes

(Mild singularities may be allowed in the background spacetime, which may introduce additional degrees of freedom.)

Such a theory has a membrane coupled to the 3-form field of 11D SUGRA, but there is no scalar field which might have been used to measure the strength of the coupling.

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Brane asymptotics

M-theory has 2-branes and 5-branes, which again are (Lorentz-signature) defects in spacetime of dimension $k+1$ ($k=2$ or 5).
Let M^{k+1} be the brane worldvolume , and let I^{10-k} denote the transverse space near the brane. The asymptotic behavior of the spacetime metric near a stack of N k-branes is a warped product

$$ds^2 = \varphi(y)dx_M^2 + dy_I^2$$

accompanied by a field strength for the SUGRA $(8-k)$-form field:

$$\star(d\varphi \wedge d\,\mathrm{vol}_M).$$

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The theories on the brane worldvolumes are quite mysterious.

Traditional approach to F-theory

F-theory is traditionally described by means of dualities.

- F-theory is the limit of M-theory when the spacetime is fibered by T^2's whose area goes to zero in the limit. From the F-theory perspective, the vacuum contains a circle whose radius goes to ∞ in the limit.
- F-theory vacua which are fibered by S^2's may have a limit in which the S^2's break in half; in this case, there is a duality with certain vacua of the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$-heterotic string.
- F-theory vacua may have a limit which is pure type IIB string theory (the "Sen limit").

In spite of the lack of a direct definition, F-theory has been a useful tool for understanding the connections between quantum gravity theories, and for model-building in particle physics.

What is F-theory?

F-theory is a ten-dimensional quantum theory of gravity semi-classcially approximated by:

- $SL(2, \mathbb{Z})$-equivariant type IIB SUGRA with quantized p-form fields, coupled to gauge theories along 7-branes and 3-branes with additional massless fields associated to brane intersections.

(Mild singularities may be allowed in the background spacetime, which may introduce additional degrees of freedom.)
The 7-branes are sources for a scalar field in the theory, and there is a richer variety of F-theory 7-branes than of string theory 7-branes.
Much can be learned about the theory by studying the D3-brane which couples to the self-dual 4-form field in the theory.

Background fields in F-theory

The background fields in F-theory consist of

- an $SL(2,\mathbb{Z})$-invariant complex scalar field with positive imaginary part
- an $SL(2,\mathbb{Z})$-doublet of 2-form fields
- a self-dual 4-form field
- the field content of the 8D and 4D coupled theories (gauge fields and scalars)
- massless fields at brane intersections
- a metric (with specified asymptotics near branes)
- fermions

This theory does not have individual strings, but rather, has an $SL(2,\mathbb{Z})$-doublet of strings.
Similarly, the theory does not have 5-branes, but rather, an $SL(2,\mathbb{Z})$-doublet of 5-branes.

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Elliptic functions and elliptic integrals

To give a good description of an $SL(2,\mathbb{Z})$-invariant scalar field, we turn to the theory of elliptic functions and elliptic integrals.

The starting point is a complex scalar τ taking values in the upper half plane, on which $SL(2,\mathbb{Z})$ acts via fractional linear transformations:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}.$$

(This is the same group action which preserves type IIB supergravity, when τ is defined properly.)

Rather than directly writing an $SL(2,\mathbb{Z})$-invariant expression, we use *modular forms of weight m*, which are functions $f(\tau)$ that transform under $\mathrm{SL}_2(\mathbb{Z})$ as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m f(\tau)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Elliptic functions and elliptic integrals

A simple way to write down some modular forms of even weight $m = 2k$ for $SL_2(\mathbb{Z})$ is to use *normalized Eisenstein series*, defined as

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(m\tau+n)^{2k}},$$

where $\zeta(2k) = \sum_{n\geq 1} n^{-2k}$ is Riemann's zeta function.

It is known that $E_4(\tau)$ and $E_6(\tau)$ freely generate the entire ring of modular forms for $SL_2(\mathbb{Z})$. The following modular form plays a special role:

$$\begin{aligned}\Delta_{12}(\tau) &= 4\left(-\frac{1}{3}E_4(\tau)\right)^3 + 27\left(\frac{2}{27}E_6(\tau)\right)^2 \\ &= -\frac{4}{27}E_4(\tau)^3 + \frac{4}{27}E_6(\tau)^2\end{aligned}$$

Elliptic functions and elliptic integrals

The key property needed for our formulation of F-theory is a result essentially due to Weierstrass: the quantities $f = -\frac{1}{3}E_4(\tau)$ and $g = \frac{2}{27}E_6(\tau)$ can be used to recover τ itself, up to $\mathrm{SL}_2(\mathbb{Z})$-transformations, provided that $4f^3 + 27g^2 \neq 0$.

Weierstrass showed this by means of *elliptic functions.* Let

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right),$$

which is a doubly-periodic meromorphic function of z.

If we set $x = -\frac{1}{\pi^2}\wp(z,\tau)$ and $y = \frac{1}{2\pi i}\frac{\partial\wp(z,\tau)}{\partial z}$, it turns out that

$$y^2 = x^3 - \frac{1}{3}E_4(\tau)\,x + \frac{2}{27}E_6(\tau).$$

Elliptic functions and elliptic integrals

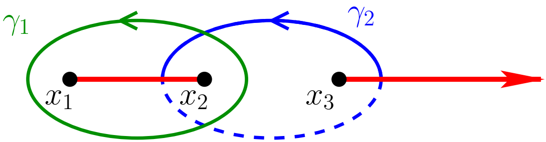
On the other hand, for any complex numbers f and g with $4f^3 + 27g^2 \neq 0$, there is an associated *elliptic integral*

$$\int_\gamma \frac{dx}{\sqrt{x^3 + fx + g}}$$

which requires the choice of a contour of integration γ to fully specify it.

To describe the choice of contour, it is convenient to consider the Riemann surface defined by the integral, which is a double cover of the complex x-plane branched at the zeros of $x^3 + fx + g$ (and at infinity). This double cover is usually depicted with the aid of two branch cuts: one from x_1 to x_2 and the other from x_3 to ∞, where $\{x_1, x_2, x_3\}$ are the roots of $x^3 + fx + g$.

Elliptic functions and elliptic integrals



A common choice of contours is: let γ_1 encircle the branch cut between x_1 and x_2, and let γ_2 cross between the branches while looping around x_2 and x_3. When these contours are oriented as shown in the figure, the ratio

$$\tau := \int_{\gamma_1} \frac{dx}{\sqrt{x^3 + fx + g}} \Bigg/ \int_{\gamma_2} \frac{dx}{\sqrt{x^3 + fx + g}}$$

lies in the upper half-plane.

Elliptic functions and elliptic integrals

The key fact is: for this τ, there is a nonzero complex number λ such that

$$E_4(\tau) = -3\lambda^4 f, \quad E_6(\tau) = \frac{27}{2}\lambda^6 g.$$

That is, the pair (f, g) up to scaling determines the scalar τ up to $\mathrm{SL}_2(\mathbb{Z})$ transformation, and *vice versa*.

Application

To apply this to F-theory, we exploit the scaling in a way that is familiar: to allow the scale factor to vary from point to point, we specify a complex line bundle $\mathcal{L}$ as part of the data of an F-theory vacuum, and specify sections

$$f \in H^0(\mathcal{L}^{\otimes 4}), \quad g \in H^0(\mathcal{L}^{\otimes 6}).$$

Then τ is implicitly defined as a ratio of elliptic integrals.

Background fields in F-theory, II

The background fields in F-theory consist of

- A complex line bundle $\mathcal{L}$ on spacetime, together with sections $f \in H^0(\mathcal{L}^{\otimes 4}), g \in H^0(\mathcal{L}^{\otimes 6})$
- an $SL(2,\mathbb{Z})$-doublet of 2-form fields
- a self-dual 4-form field
- 7-branes located at $\{4f^3 + 27g^2 = 0\}$
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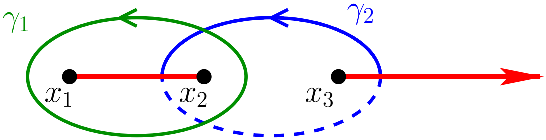
The $SL_2(\mathbb{Z})$-doublet of 2-form fields

(cf. Douglas–Park–Schnell arXiv:1403.1595)



Returning to the choice of contours for the elliptic integrals, it is not hard to show that any closed contour γ is homotopic to $p\gamma_1 + q\gamma_2$ for $p, q \in \mathbb{Z}$. Thus, at each point in spacetime with $4f^3 + 27g^2 \neq 0$, the set of integration contours gives a copy of $\mathbb{Z}^2$, and the $SL_2(\mathbb{Z})$-ambiguity in τ is precisely tracked by the $SL_2(\mathbb{Z})$ action on $\mathbb{Z}^2$.

In fancy language, these $\mathbb{Z}^2$'s together form a *sheaf of contours* $\mathcal{C}_\mathbb{Z}$, and we may use the transformation laws of this sheaf to form a sheaf of doublet 2-forms

$$\mathcal{C}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{A}^2$$

(where $\mathcal{A}^2$ denotes the sheaf of 2-forms on spacetime).

The $\mathrm{SL}_2(\mathbb{Z})$-doublet of 2-form fields

Since 2-forms couple to strings, the sheaf $\mathcal{C}_{\mathbb{Z}}$ may also be interpreted as the *sheaf of local string charges* of the theory. On any open set U in spacetime over which the sheaf $\mathcal{C}_{\mathbb{Z}}$ is trivial, individual strings can be defined. But if the sheaf $\mathcal{C}_{\mathbb{Z}}$ is globally non-trivial on spacetime, individual strings cannot be defined.

A background doublet two-form field is just a section of the sheaf $\mathcal{C}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{A}^2$, i.e., a locally-defined pair of two-forms labeled by the local string charges.

Background fields in F-theory, III

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Another appearance of $\mathrm{SL}_2(\mathbb{Z})$

We are describing the $\mathrm{SL}_2(\mathbb{Z})$-invariant scalar in F-theory by means of the "Weierstrass coefficients" f and g, which are sections of related line bundles. Topologically, a family of such objects is determined by a map to an appropriate classifying space, which in this case is

$$\mathcal{M} := \{(f, g) \in \mathcal{E}^{\otimes 4} \oplus \mathcal{E}^{\otimes 6} \mid 4f^3 + 27g^2 \neq 0\},$$

where $\mathcal{E}$ is the total space of the universal bundle over $\mathbb{CP}^\infty$. A 7-brane in the theory will source the scalar field in some way, specified by a loop in $\mathcal{M}$. The essential nature of this sourcing is topological, and only the homotopy class of the loop matters. Here is another remarkable fact:

$$\pi_1(\mathcal{M}) \cong \mathrm{SL}_2(\mathbb{Z}).$$

Although there are a number of related facts in the mathematics literature, this particular statement doesn't seem to have been observed before. The analogous statement in algebraic geometry (computing the fundamental group of the moduli stack) was proved by B. Noohi

Another appearance of $SL_2(\mathbb{Z})$

This is strong evidence in favor of this model of F-theory, for it has an important feature: $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ corresponds to a nontrivial brane, even though the corresponding action on τ is trivial. That is, for the previous discussion of modular forms, the group $PSL_2(\mathbb{Z})$ would have sufficed.

A catalog of 7-branes

- If the $\mathrm{SL}_2(\mathbb{Z})$ transformation has a fixed element when acting on $\mathbb{Z}^2$, there is a well-defined string charge near the brane. This has a standard description as a stack of N D7-branes with $\mathfrak{u}(N)$ gauge field (or $\mathfrak{su}(N)$ ...)
- If not, a small perturbation of (f, g) may realize the brane as a concatenation of other, mutually non-local, branes.
  - Concatentation: Gaberdiel-Zwiebach.
  - Systematic construction of Lie algebra via string junctions: DeWolfe-Zwiebach.
  - Realization as small perturbation of (f, g): Grassi–Halverson–Shaneson arXiv:1306.1832
- Use three generators $A = T^{-1}$ with string charge $(1, 0)$, $B = ST^2$ with string charge $(-1, 1)$, and $C = T^2S$ with string charge $(1, 1)$.
  (Here, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.)


D7

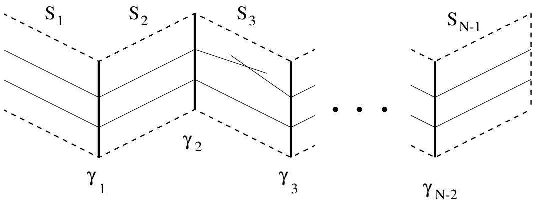
A catalog of 7-branes

type	$SL_2(\mathbb{Z})$ class	perturbation	algebra
$I_{n},\ n\geq 1$	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	A^n	$\mathfrak{su}(n)$ or $\mathfrak{sp}([\frac{n}{2}])$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	AC	–
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	A^2C	$\mathfrak{su}(2)$
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	A^3C	$\mathfrak{su}(3)$ or $\mathfrak{sp}(1)$
I_0^*	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	A^4BC	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$
$I_n^*,\ n\geq 1$	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	$A^{n+4}BC$	$\mathfrak{so}(2n+8)$ or $\mathfrak{so}(2n+7)$
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	A^5BC^2	$\mathfrak{e}_6$ or $\mathfrak{f}_4$
III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	A^6BC^2	$\mathfrak{e}_7$
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	A^7BC^2	$\mathfrak{e}_8$

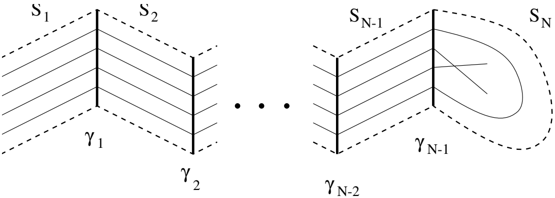
Additional monodromy

There is additional monodromy on the conjugacy class, which determines the actual algebra.
This can be seen by studying the junction algebras from the A, B, C description, or from the resolution graphs in M-theory and the Kodaira classification of singular fibers. I illustrate the latter approach:

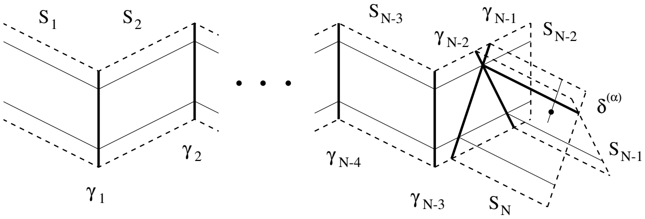
$\mathfrak{su}(N)$:



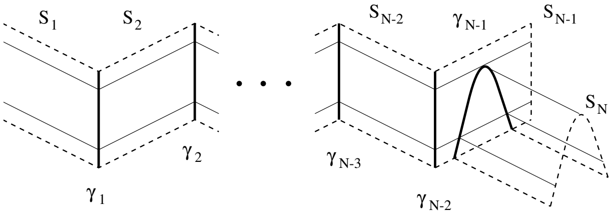
$\mathfrak{sp}(2N)$:



$\mathfrak{so}(2N)$:



$\mathfrak{so}(2N+1)$:



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- a section of the sheaf $\mathcal{C}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{A}^2$, where $\mathcal{C}_\mathbb{Z}$ is the sheaf of possible integration contours for the elliptic integral $\int_\gamma \frac{dx}{\sqrt{x^3+fx+g}}$, i.e., the sheaf of local string charges
- a self-dual 4-form field
- 7-branes located at $\{4f^3 + 27g^2 = 0\}$, as per the catalog, with a specified orientation of the normal bundle
- the field content of the 8D and 4D coupled theories (gauge fields and scalars)
- massless fields at brane intersections
- a metric (with specified asymptotics near branes)
- fermions

Background fields in F-theory, IV

The background fields in F-theory consist of

- A complex line bundle $\mathcal{L}$ on spacetime, together with sections $f \in H^0(\mathcal{L}^{\otimes 4}), g \in H^0(\mathcal{L}^{\otimes 6})$
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Further directions

As mentioned, there are several directions in which this treatment of F-theory should be expanded.

- Include $\overline{D7}$-branes.
- Include webs of (p, q) 5-branes. Perhaps also webs of (p, q) 1-branes.
- Specify the conditions for supersymmetry.