The global gravitational anomaly of the self-dual field theory

Samuel Monnier

Laboratoire de Physique Théorique, Ecole Normale Supérieure

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MOTIVATION

The self-dual field theory is the theory of an abelian 2ℓ-form gauge field, living on a $4\ell + 2$-dimensional manifold, whose $2\ell + 1$-form field strength obey a self-duality condition: $F = *F$.

Examples:

- The chiral boson in two dimensions.
- The world volume theory of the M-theory and IIA fivebranes.
- The chiral two-forms in 6d supergravities.
- The RR 4-form gauge field in type IIB supergravity.

The global gravitational anomaly of this theory is still unknown. Witten made a proposal, in the case when the self-dual field has no zero modes. How to get a general formula?

Plan of the talk

1. Gravitational anomalies
2. The self-dual field
3. The anomaly bundle
4. Derivation of the global anomaly formula
5. Type IIB supergravity
6. Outlook

GRAVITATIONAL ANOMALIES

Gravitational anomalies

Given an Euclidean QFT, put it on a compact Riemannian manifold M.

The metric g can be seen as an external parameter, on which the partition function Z depends.

Under the action of a diffeomorphism ϕ, $Z(g)$ is not necessarily invariant. In general

$$Z((\phi^{-1})^* g) = \xi(\phi, g) Z(g) \quad \xi(\phi, g) \in \mathbb{C}$$

Consistency with the group structure requires

$$\xi(\phi_2 \circ \phi_1, g) = \xi(\phi_2, (\phi_1^{-1})^* g) \xi(\phi_1, g)$$

ξ is a 1-cocycle for the group $\mathcal{D}$ of diffeomorphisms of M.

Gravitational anomalies

A 1-cocycle such as ξ defines a line bundle $\mathscr{A}$ over $\mathcal{M}/\mathcal{D}$, the anomaly bundle.

The anomaly bundle carries a connection $\nabla_{\mathscr{A}}$.

In general, the partition function is not a well-defined function on $\mathcal{M}/\mathcal{D}$, but rather a section of $\mathscr{A}$.

The *local anomaly* is the curvature of $\nabla_{\mathscr{A}}$.

The *global anomaly* is the set of holonomies of $\nabla_{\mathscr{A}}$.

Gravitational anomalies

Why are we interested in anomalies?

To couple the theory to gravity, we have integrate $Z(g)$ over $\mathcal{M}/\mathcal{D}$.

This is possible only if $Z(g)$ is an honest function over $\mathcal{M}/\mathcal{D}$.

In other words, this is possible only if the local and global anomalies vanish.

$\Rightarrow$ Any low energy effective action obtained from a supposedly consistent quantum gravity theory must have vanishing gravitational anomalies.

$\Rightarrow$ Strong constraints on low energy effective actions in cases where anomalies can arise.

Gravitational anomalies

Gravitational anomalies occur typically in chiral fermionic theories in dimension $4\ell + 2$.

Consider the fibration $(M \times \mathcal{M})/\mathcal{D}$ over $\mathcal{M}/\mathcal{D}$ with fiber M.

A chiral fermionic theory produces a family of chiral Dirac operators on the fibers.

The functional determinant of a chiral Dirac operator is the section of a line bundle $\mathscr{D}$ with connection $\nabla_{\mathscr{D}}$ over $\mathcal{M}/\mathcal{D}$.

The anomaly bundle $\mathscr{A}$ of the chiral fermionic theory coincides with $\mathscr{D}$, as a line bundle with connection.

Gravitational anomalies

Using index theory techniques, Bismut and Freed proved general formulas for the curvature and holonomies of $\nabla_{\mathscr{D}}$.

They recovered and generalized expressions obtained by Alvarez-Gaumé and Witten (local anomaly) and Witten (global anomaly).

Local anomaly:

$$R_{\mathscr{D}} = \left[2\pi i \int_M \hat{A}(R_{TM}) \mathrm{ch}(R_{\mathscr{E}}) \right]^{(2)}$$

$$\hat{A}(R) = \sqrt{\det \frac{R/4\pi}{\sinh R/4\pi}} \,, \quad \mathrm{ch}(R) = \mathrm{Tr} \exp iR/2\pi \ .$$

Gravitational anomalies

For the global anomaly:

1. Pick a cycle c in $\mathcal{M}/\mathcal{D}$.
2. Construct the mapping torus $\hat{M}_c$. Fibration over c with fiber M.
3. Pick a metric g_c on c and set $g_\epsilon = g_c/\epsilon^2 \oplus g_M$, a metric on $\hat{M}_c$.
4. Consider the Dirac operator $\hat{D}_\epsilon$ on $\hat{M}_c$ twisted by $\mathscr{E}$.
5. Let η_ϵ be its eta invariant and h_ϵ the dimension of its space of zero modes.
6. The holonomy is

$$\mathrm{hol}_{\mathscr{D}}(c) = (-1)^{\mathrm{index} D} \lim_{\epsilon \to 0} \exp -\pi i(\eta_\epsilon + h_\epsilon)$$

Gravitational anomalies

Useless formula, because the eta invariant is impossible to compute explicitly.

However, if $k \cdot \hat{M}_c$ is bounded by a spin manifold W, one can use the Atiyah-Patodi-Singer theorem to obtain a useful formula, of the form

$$\frac{1}{2\pi i} \ln \mathrm{hol}_{\mathscr{D}}(c) = \frac{1}{k}\left(\mathrm{index}A - \int_W \hat{A}(R_{TM})\mathrm{ch}(R_{\mathscr{E}})\right)$$

If we know that the local anomaly vanishes, the integral terms cancel.

$\Rightarrow$ One only has to check that a sum of topological invariants is an integer.

Gravitational anomalies

This solves the problem of computing gravitational anomalies for chiral fermionic theories.

But the self-dual field theory is a chiral theory which does not fall into this framework.

Its local anomaly can be described with the signature Dirac operator.

What about the global anomaly?

This is a gap in our understanding of effective field theories coming from quantum gravity.

GRAVITATIONAL ANOMALIES

Applications:

- Check the consistency of supergravities containing self-dual fields:
  - Type IIB
  - Six dimensional supergravities with $T \neq 1$. (Applications to the six-dimensional landscape.)
- Check anomaly cancellation in IIA and M-theory backgrounds containing five-branes.
- Understand better the contribution of five-brane instantons to four and three dimensional effective actions.

THE SELF-DUAL FIELD

THE SELF-DUAL FIELD

To compute the global anomaly, we need a global definition (on $\mathcal{M}/\mathcal{D}$) of the quantum self-dual field theory on an arbitrary Riemannian manifold. $\rightarrow$ Non-trivial problem.

Known ways of constructing the partition function

- Holomorphic factorization of the abelian 2ℓ-form gauge field partition function.
- Geometric quantization of $H^{2\ell+1}(M, U(1))$.

None of these techniques give information about the global gravitational anomaly.

It seems that only path integration of an action can provide a global definition of the partition function.

The self-dual field

There exists many classical actions for the self-dual field. But it is not clear if the quantum partition function can be constructed from the naive path integration of a Gaussian action.

Indeed, it is not even clear what the off-shell degrees of freedom of the self-dual field should be on an arbitrary Riemannian manifold.

	Abelian gauge field	Self-dual field
On-shell	$H^{2\ell+1}(M,\mathbb{R})$	$H^{2\ell+1}_{SD}(M,\mathbb{R})$
Off-shell	$\Omega^{2\ell+1}_{\text{ex}}(M)$	???

The self-dual field

Can we really define the self-dual field theory as a free Gaussian field theory? How to quantize the existing actions? How to define the theory on an arbitrary Riemannian manifold?

Fortunately, one can write a Gaussian action principle for *a pair* of self-dual fields.

$\Rightarrow$ We can construct the square of the partition function.

This provides almost complete information about the global anomaly of a single self-dual field. Only a mild ambiguity will remain in the end.

The self-dual field

Let M be a manifold of dimension $4\ell + 2$.

We endow it with a quadratic refinement of the intersection form. Parameterized by $\eta \in \frac{1}{2}H^{2\ell+1}_{\mathrm{free}}(M, \mathbb{Z})/H^{2\ell+1}_{\mathrm{free}}(M, \mathbb{Z})$.

We write $\mathcal{D}_\eta$ the group of diffeomorphisms of M preserving the quadratic refinement.

We want to construct and study the partition function of a pair of self-dual fields over $\mathcal{M}/\mathcal{D}_\eta$.

The self-dual field

For manifolds of dimension $4\ell + 2$:

- The intersection product gives a symplectic structure ω on $\Omega^{2\ell+1}(M)$.
- The Hodge star operator squares to $-\mathbb{1}$ on $\Omega^{2\ell+1}(M)$, hence defines a complex structure.
- Both structures restrict to $H^{2\ell+1}(M, \mathbb{R})$

A complex structure on a $2n$-dimensional symplectic vector space such as $H^{2\ell+1}(M, \mathbb{R})$ can be parameterized by a complex $n \times n$ matrix τ with positive definite imaginary part. Siegel upper-half plane $\mathcal{C}$.

The self-dual field

Partition function:

$$\mathcal{Z}(Z,\eta) = u^{-1}(g_0)\,\det(-i\tau_+)^{-1/2}(\theta^\eta(Z,\tau))^2$$

$$u(g) = \prod_{p=0}^{2\ell}\left(\left(\mathrm{Vol}(H^p)^{-2}\det(d^\dagger d|_{\Omega^p_{\mathrm{coex}}})\right)^{(-1)^p}\right)^{1/2}$$

Properties:

- Holomorphic (self-duality)
- Its norm is compatible with results from geometric quantization and holography.
- Compatible with the known local anomaly of the self-dual field theory.

The anomaly bundle

The anomaly bundle

We saw that Dirac operators allow to construct line bundles over $\mathcal{M}/\mathcal{D}_\eta$. Here is another construction.

The action of $\mathcal{D}_\eta$ on M induces an action on $H^{2\ell+1}(M, \mathbb{R})$.

This action is symplectic with respect to ω, preserves the integral cohomology and factors through $\Gamma_\eta \subset \mathrm{Sp}(2n, \mathbb{Z})$.

$\Rightarrow$ We have a map $\mathcal{M}/\mathcal{D}_\eta \to \mathcal{C}/\Gamma_\eta$.

Bundles over $\mathcal{C}/\Gamma_\eta$:

- Theta bundle $\mathscr{C}^\eta$
- Determinant bundle $\mathscr{K}$ of the Hodge bundle

These bundles can be pulled back to $\mathcal{M}/\mathcal{D}_\eta$.

The anomaly bundle

Call $\mathscr{D}$ the determinant bundle of the Dirac operator coupled to chiral spinors, and $\mathscr{D}_s$ the determinant bundle of the signature operator.

Some relations:

- $\mathscr{D}_s = \mathscr{D}^2$.
- $\mathscr{D} \simeq (\mathscr{K})^{-1}$
- $\mathscr{F}^\eta := (\mathscr{C}^\eta)^2 \otimes (\mathscr{K})^{-1}$ is a flat bundle. Its holonomies can be computed explicitly from the transformation formula of the Siegel theta functions. Character χ^η of Γ_η.

The anomaly bundle

The partition function of a pair of self-dual fields factorizes:

$$\mathcal{Z} = (\theta^\eta)^2 \cdot (\text{one loop determinant})$$

The one-loop determinant vanishes *nowhere* on $\mathcal{M}/\mathcal{D}_\eta$! $\Rightarrow$ It is the section of a topologically trivial bundle.

The squared theta function is a section of $(\mathscr{C}^\eta)^2$.

$\Rightarrow$ Topologically, the anomaly bundle of a pair of self-dual fields $(\mathscr{A}^\eta)^2$ is $(\mathscr{C}^\eta)^2$.

The anomaly bundle

It has been known for a long time that the local anomaly of a pair of self-dual field is described correctly by $\mathscr{D}^{-1}$.

$\Rightarrow$ The curvatures of the connections on $(\mathscr{A}^\eta)^2$ and on $\mathscr{D}^{-1}$ have the same local form.

$\Rightarrow$ $(\mathscr{A}^\eta)^2$ and $\mathscr{D}^{-1}$ coincide up to a flat bundle.

As $(\mathscr{A}^\eta)^2 \simeq (\mathscr{C}^\eta)^2$ topologically, we have

$$(\mathscr{A}^\eta)^2 = \mathscr{D}^{-1} \otimes \mathscr{F}^\eta$$

$$\Big(= \mathscr{D}^{-1} \otimes (\mathscr{K})^{-1} \otimes (\mathscr{C}^\eta)^2 \Big)$$

THE ANOMALY BUNDLE

We determined the anomaly bundle for a pair of self-dual fields and its connection.

How to compute the global anomaly?

- The holonomies of the connection on $\mathscr{D}^{-1}$ are provided by the Bismut-Freed formula.
- The holonomies of $\mathscr{F}^{\eta}$ known from the theta transformation formula.

$\Rightarrow$ Problem solved?

No! The resulting holonomy formula is practically useless. We need to reexpress the holonomies in term of topological invariant of a manifold bounding the mapping torus.

THE GLOBAL ANOMALY FORMULA

The global anomaly formula

To relate the eta invariant in the Bismut-Freed formula to data on a bounding manifold, we have to use the Atiyah-Patodi-Singer theorem.

Impossible with $D \Rightarrow$ Consider D_s instead:

$$(\mathscr{A}^\eta)^4 = \mathscr{D}_s^{-1} \otimes (\mathscr{F}^\eta)^2$$

$$\mathrm{hol}_{(\mathscr{A}^\eta)^4}(c) = (\chi^\eta(\gamma_c))^2 \exp \pi i(\eta_0 + h)$$

It turns out that $(\chi^\eta(\gamma_c))^2 \exp \pi i h = \exp -\pi i A_\eta$, where A_η is the Arf invariant of the mapping torus $\hat{M}_c$.

The global anomaly formula

$$\mathrm{hol}_{(\mathscr{A}^{\eta})^4}(c) = \exp \pi i(\eta_0 - A_\eta)$$

APS says $\eta_0 = \int_W L - \sigma_W$.

For suitable W, $A_\eta = \int_W \lambda^2 - \sigma_W \bmod 8$. λ a Wu class for the intersection pairing on W.

$$\mathrm{hol}_{(\mathscr{A}^{\eta})^4}(c) = \exp \pi i \int_W (L - \lambda^2)$$

We need to take a fourth root of this formula. Non-trivial operation!

The global anomaly formula

Most naive way of taking the fourth root:

$$\mathrm{hol}_{\mathscr{A}^\eta}(c) = \exp \frac{2\pi i}{8} \int_W (L - \lambda^2)$$

Consistency checks:

- Reproduces the correct relative anomaly for $\mathscr{A}^{\eta'} \otimes (\mathscr{A}^\eta)^{-1}$ (Lee-Miller-Weintraub).
- If the formula above defines the holonomies of a well-defined bundle, then it is the correct one, by an argument on $\mathcal{C}/\Gamma_\eta$.
- This is the case when $\lambda = 0$.
- The formula appeared in the purely mathematical, but supposedly related work of Hopkins and Singer.

Type IIB supergravity

Type IIB supergravity

The anomaly formula is useful only if we can determine λ. $\Rightarrow$ Applications still require some work.

In the case of 10-dimensional type IIB supergravity, λ can be taken to vanish.

“Vanilla” IIB supergravity: the topological sectors of the Ramond-Ramond fields are labelled by cohomology, not K-theory.

Type IIB supergravity

Global anomaly cancellation in Type IIB has been already studied by Witten in his original paper on global gravitational anomalies.

He used:

$$\mathrm{hol}_{\mathscr{A}}(c) = \exp \frac{2\pi i}{8} \left(\int_W L - \sigma_W \right)$$

and obtained $\exp -\frac{2\pi i}{8}\sigma_W$ for the total global anomaly.

Would type IIB suffer from a global anomaly?

Witten showed that it's not the case in 10d Minkowsky space-time.

He also mentionned that his result should be trusted only when $H^{2\ell+1}(M, \mathbb{Z}) = 0$.

Indeed in this case $(\mathcal{A})^4 = (\mathcal{D}_s)^{-1}$.

Type IIB supergravity

As $\lambda = 0$, our anomaly formula predicts

$$\mathrm{hol}_{\mathscr{A}}(c) = \exp \frac{2\pi i}{8} \int_W L$$

$\Rightarrow$ No gravitational anomaly.

This is compatible with Witten's result, because σ_W is a multiple of 8 whenever $H^{2\ell+1}(M, \mathbb{Z}) = 0$.

$\Rightarrow$ In conclusion:

- This (slightly naive) check shows that there is no global gravitational anomaly in type IIB supergravity.
- Witten's formula is valid when $H^{2\ell+1}(M, \mathbb{Z}) = 0$, but does not make the anomaly cancellation manifest.

Conclusions

We derived the global gravitational anomaly of the self-dual field from first principle (i.e. from an action principle).

There is a lot of work ahead:

- Check anomaly cancellation in IIB taking into account K-theory.
- Derive an anomaly formula for the five-branes.
- Check anomaly cancellation in 6d supergravities.

The main difficulty is to determine the class λ.

Given all we already learned from anomalies about supergravities, string theory and M-theory, we can hope global anomalies will provide new insights.