Supersymmetric Geometries of M-Theory

[DM, J. Sparks] hep-th/0306225
Motivations

- M-Theory compactifications with fluxes

- It is interesting to study models with $\mathcal{N} = 1$ in $d = 3$ (e.g. in relation to the cosmological constant problem)

- String/M-Theory compactifications to four dimensions

- Holography

- AdS solutions dual to SCFT

- RG flow solutions

- Relation to interesting holographic flows in String Theory (e.g. Polchinski–Strassler, Klebanov–Strassler)

- Geometric conditions corresponding to M5-brane wrapping various cycles (especially ‘exotic’ ones, e.g. associative)

- The same conditions are satisfied by particular brane intersections

- Emphasize the role of $G$-structures and especially of generalized calibrations in the context of supersymmetric geometries with fluxes
Killing spinor equations, special holonomy, and G-structures

Supersymmetric solutions of supergravity admit ‘Killing spinors’

Killing spinor equations

\[ \delta \psi_M = \nabla_M \epsilon + \Omega_M \epsilon = 0 \]

\(+\) possible algebraic equations for \( \epsilon \): \( O \epsilon = 0 \)

\# independent solutions \( \iff \) unbroken supersymmetry

When all the fields are set to zero, except the metric, solutions to \( \nabla_M \epsilon = 0 \) are Ricci-flat, special holonomy manifolds:

\[
\begin{align*}
\text{dim}(M) &= 2n & SU(n) \text{ (Calabi–Yau)} \\
\text{dim}(M) &= 4n & Sp(n) \text{ (hyper–Kähler)} \\
\text{dim}(M) &= 7 & G_2 \\
\text{dim}(M) &= 8 & Spin(7)
\end{align*}
\]

• Including the form-fields (i.e. Fluxes) \( \rightarrow \) \( \Omega_M \neq 0 \)

\( \rightarrow \) the holonomy of the Levi–Civita (or spin-) connection is not a good principle for determining the number of solutions

\( \blacktriangleright \) **G-structures** are the most appropriate mathematical framework to study geometries with fluxes [Gauntlett, DM, Pakis, Waldram]
A $G$-structure on a manifold $M$ is a (global) reduction of the frame bundle to a sub-bundle with fibre $G$.

**Key facts:**

- equivalent to a set of (globally defined) $G$-invariant tensors, or spinors, on $M$
- tensors are decomposed into irreducible representations of $G$
- departure from special holonomy is measured by the **Intrinsic Torsion**

$$T \in \Lambda^1 \otimes g^\perp = \bigoplus_{i=1}^{n} \mathcal{W}_i, \quad g \oplus g^\perp = \text{so}(n)$$

Each class is characterized by calculable quantities, denoted $\mathcal{W}_i$

- Why is this useful in supergravity?

**Solutions to the Killing spinor equations define a particular $G$-structure, and its type ("class of intrinsic torsion") can be determined analyzing bosonic equations obeyed by the $G$-invariant forms**

**Flux = Intrinsic Torsion**
Example: $G_2$-structures in $d = 7$ [Fernandez,Gray]

- These are defined by the associative three-form $\phi$
- Under $SO(7) \to G_2$, $21 \to 7 + 14$
- Classes of intrinsic torsion

\[ T \in \Lambda^1 \otimes g_2^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \]
\[ 7 \times 7 \to 1 + 14 + 27 + 7 \]

\[ d\phi \in \Lambda^4 \cong \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \]
\[ 35 \to 1 + 27 + 7 \]

\[ d \ast \phi \in \Lambda^5 \cong \mathcal{W}_2 \oplus \mathcal{W}_4 \]
\[ 21 \to 14 + 7 \]

E.g. $M$ has $G_2$-holonomy iff $d\phi = 0 = d \ast \phi$
M-Theory on eight-manifolds with fluxes \([\text{DM,Sparks}]\)

- **Aim**: characterize the most general supersymmetric M-Theory "compactifications" to three-dimensions

- 11-dim supergravity

Bosonic fields: \(g_{MN}, G_{MNPQ}\)

Killing spinor equation:

\[
\delta \psi_M = \hat{\nabla}_M \eta - \frac{1}{288} \left( G_{NPQR} \hat{\nabla}^{NPQR} M - 8G_{MNPQ} \hat{\nabla}^{NPQ} \right) \eta
\]

Bianchi identity: \(\text{d}G = 0\)

\(G\) eq. of motion: \(\text{d} \hat{*} G + \frac{1}{2} G \wedge G = X_8 \rightarrow \chi(M_8) \sim \int G \wedge G\)

- We are interested in compactifications to \(d = 3\) which preserve (at least) \(N = 1\) supersymmetry.

warped metric \(ds_{11}^2 = e^{2\Delta(x)} (ds_3^2 + g_{mn} dx^m dx^n)\)

\(\mathbb{R}^{1,2}\) or \(\text{AdS}_3\) no ansatz

flux \(G_{mnpq}\) arbitrary, \(G_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\sigma\tau} f_{\tau}\)

spinor \(\eta = \psi \otimes \xi\)

Note: we have made no particular assumptions, apart from requiring the 3-dim external space to be Minkowski\(_3\) or AdS\(_3\)
Reminder of known results [K. Becker], [Acharya, de la Ossa, Gukov]

• Assumption: $\xi$ is a $\text{Spin}(8)$ Majorana–Weyl spinor of definite chirality

$\Rightarrow$ It defines a $\text{Spin}(7)$-structure on $M_8$. Equivalently defined by the $\text{Spin}(7)$-invariant Cayley four-form

$$\Psi_{mnpq} = \xi^T \gamma_{mnpq} \xi$$

• General solution

$$d\tilde{s}^2_{11} = H^{-2/3}d\tilde{s}^2(\mathbb{R}^{1,2}) + H^{1/3}d\tilde{s}^2(\text{Spin}(7))$$

$$G = \text{vol}_3 \wedge d(H^{-1}) + G_{27}$$

$$\tilde{\ast} \Box H + \frac{1}{2} G_{27} \wedge G_{27} = X_8$$

$G_{27}$ is in the 27 representation of the $\text{SO}(8) \rightarrow \text{Spin}(7)$ decomposition of four-forms: $70 \rightarrow 35 + 27 + 7 + 1$

• These are clearly M2-brane-type of solutions

• AdS$_3$ compactifications are ruled out

• Additional Killing spinors of the same chirality reduce further the holonomy of $d\tilde{s}^2$. E.g. $\mathcal{N} = 2 \rightarrow SU(4)$ [K. Becker, M. Becker], $\mathcal{N} = 3 \rightarrow Sp(2)$, etc.
Motivations for the existence of more general solutions

- More general Minkowski\(_3\) vacua from wrapped M5-branes

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

M5-branes wrapped on \(G_2\)

associative 3-cycles inside \(G_2 \times S^1 \to \mathcal{N} = 1\)

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

M5-branes wrapped on SLAG

3-cycles inside \(CY_3 \times T^2 \to \mathcal{N} = 2\)

- (Warped) AdS\(_3\) vacua

M5-branes wrapped on 4-cycles or M2-branes wrapped on \(S^1 \to\) effective strings. In the ‘near-horizon’ limit should give AdS\(_3\) vacua

Consider simply the Freund–Rubin solutions: \(AdS_4 \times M_7\) with \(M_7\) having weak \(G_2\) holonomy, and write the metric as

\[
d\tilde{s}_{11}^2 = \cosh^2(2mr) \, ds^2(\text{AdS}_3) + dr^2 + d\tilde{s}_7^2
\]

- M5/M2 bound states

M5-brane world volume action comprises a self-dual three-form \(H\), inducing an M2-brane charge via a WZ coupling → supersymmetric solutions corresponding to (‘dyonic’ or ‘dielectric’) M5/M2

- There were known supersymmetric solutions with \(G\)-flux, which were not of the type just reviewed
• Require the existence of a $G_2$-structure defined on $M_8$, or equivalently, of a non-chiral $\text{Spin}(8)$ Killing spinor

• We take $\eta = e^{-\Delta/2} \psi \otimes (\xi_+ \oplus \xi_-)$ \hspace{1cm} $\Gamma_9 \xi_\pm = \pm \xi_\pm$

▶ This is a completely generic spinor

▶ Note that we have $\mathcal{N} = 1$ in $d = 3$: $\nabla_\mu \psi + m \gamma_\mu \psi = 0$

The $G_2$-structure is defined by the following forms:

$$K_m = \frac{1}{||\xi_+|| \cdot ||\xi_-||} \xi_+^T \gamma_m \xi_-$$

$$\phi_{mnp} = \frac{1}{||\xi_+|| \cdot ||\xi_-||} \xi_+^T \gamma_{mnp} \xi_-$$

or equivalently by two Cayley forms with opposite dualities

$$\psi_{mnpr}^\pm = \frac{1}{||\xi_\pm||^2} \xi_\pm^T \gamma_{mnpr} \xi_\pm$$

• The spinors $\xi_\pm$ cannot be normalized to unity! Rather, supersymmetry implies

$$\frac{1}{2} (||\xi_+||^2 + ||\xi_-||^2) = 1$$

$$\Rightarrow \frac{1}{2} (||\xi_+||^2 - ||\xi_-||^2) \equiv \sin \zeta$$
The geometry on the eight-manifold

- We obtain a set of conditions on the forms, equivalent to the supersymmetry conditions

- The metric has the canonical form:
  \[ d\tilde{s}_8^2 = e^{2\Delta}(g_{ij}(x,y)dx^i dx^j + e^{-6\Delta} \sec^2 \zeta dy^2) \]

- At any fixed \( y \), the \( G_2 \)-structure has intrinsic torsion in \( \mathcal{W}_3 \oplus \mathcal{W}_4 \) \( (27+7) \)

- The fluxes are completely fixed:
  \[ f = e^{-3\Delta} d(e^{3\Delta} \sin \zeta) + 4mK \cos \zeta \]
  \[ F \sin \zeta - \ast F = e^{-6\Delta} d(e^{6\Delta} \phi \cos \zeta) - 4m(i_K \ast \phi - \phi \wedge K \sin \zeta) \]

- The \( G \) equation of motion:
  \[ d(e^{6\Delta} \ast f) + \frac{1}{2} e^{6\Delta} F \wedge F = X_8 \]

The total \( G \)-flux is defined as \( G = e^{3\Delta}(F + vol_3 \wedge f) \)

- Note that \( m \neq 0 \rightarrow AdS_3 \) solutions are not ruled out

- \( \sin \zeta = \pm 1 \rightarrow \) recover M2-brane \( \perp Spin(7) \) manifold (one spinor vanishes in this limit! And \( m = 0 \))

In the following I will illustrate some examples, for which the general equations simplify
Example 1: \( F \) is self-dual

- Imposing \( F = *F \) (and \( m = 0 \)) the general equations simplify considerably
- \( e^{-3\Delta} = 1 - \sin \zeta \)
- \( d(e^6\Delta \psi) = 0 \Rightarrow M_8 \) is to conformal to \( Spin(7) \)-holonomy

- The fluxes are
  \[
  f = 3d\Delta \\
  G_{\text{internal}} = -d(e^6\Delta \cos \zeta \phi)
  \]

- The Bianchi identity and \( G \) equation of motion are satisfied automatically \( \checkmark \)

E. g. The "deformed M2-brane solutions" obtained in [Cvetic,Lü,Pope] are of this type (the case of \( M_8 \propto \mathbb{R}^8 \) has been checked explicitly in [K.Becker,M.Becker,Sriharsha])

\[
G_{\text{internal}} = dx^{1234} \wedge dx^{5678} \\
e^{-3\Delta} = c - \mu \delta_{ij} x^i x^j
\]
Example 2: Vanishing internal flux $F$ (purely electric solutions)

- Imposing $F = 0$ the equations simplify again. Here we have:
  - $e^{-\Delta} = \cos \zeta$

- $d(e^{3\Delta \phi}) = 4m e^{4\Delta} i_K \star \phi \Rightarrow \text{(def: } \tilde{\phi} = e^{-3\Delta \phi})$

  $$ ds_{11}^2 = \sec^2 \zeta \left( ds_{3}^2(\text{AdS}_3) + \frac{1}{4m^2} d\zeta^2 \right) + ds_{7}^2 $$

- This is just $\text{AdS}_4 \times M_7$. With $M_7$ having weak $G_2$-holonomy
Wrapped or intersected M5-branes (and associated non-linear PDE's)

- In [Fayyazuddin, Smith] the supersymmetry conditions describing M5-branes wrapped on holomorphic cycles (or M5-branes intersections) were used to obtain non-linear PDS's. E.g.

\[
\partial \bar{\partial} \sqrt{g_4} + \Box_{\mathbb{R}^3} J = \text{sources}
\]

This is the Bianchi identity (dG = sources) for the geometry – a type of SU(2)-structure in \(d = 7\).

- In our approach these equations are reproduced as special cases, and generalizations to other cycles are straightforward. E.g.

**Example 3a:**

M5 wrapped on associative 3-cycles – \(\mathcal{N} = 1\) in \(d = 3\)

\[
d_7 \left[ e^{-6\Delta} \ast_7 d_7(e^{6\Delta} \phi) \right] + \Box_{\mathbb{R}}(e^{6\Delta} \ast_7 \phi) = \text{sources}
\]

**Example 3b:**

M5 wrapped on SLAG 3-cycles – \(\mathcal{N} = 2\) in \(d = 3\)

\[
d_6 \left[ e^{-9\Delta} \ast_6 d_6(e^{6\Delta} \text{Im}\Omega) \right] + \Box_{\mathbb{R}^2}(e^{3\Delta} \text{Re}\Omega) = \text{sources}
\]
Example 4: A ‘dyonic’ solution (physical significance of \( \sin \zeta \))

- M5/M2 bound state solution (in flat space) 
  [Izquierdo, Lambert, Papadopoulos, Townsend]

From the general conditions, inserting a simple ansatz, we obtain:

\[
\begin{align*}
ds^2_{11} &= H^{-\frac{3}{2}}(\sin^2 \alpha + H \cos^2 \alpha)^{\frac{1}{2}} \left[ ds^2(\mathbb{R}^{1,2}) + \frac{H}{\sin^2 \alpha + H \cos^2 \alpha} \, du \, du + H \, dx \, dx \right] \\
\tan^2 \zeta &= \frac{1}{H} \tan^2 \alpha \quad \alpha \text{ is a constant}
\end{align*}
\]

Flux: \( G = \frac{1}{2} \cos \alpha \delta_5 H + \frac{1}{2} \sin \alpha dH^{-1} \wedge \text{vol}(\mathbb{R}^{1,2}) + \sin 2\alpha (\ldots) \)

\( G \) equation of motion \( \Rightarrow \Box_{\mathbb{R}^5} H = 0 \)

- This solution interpolates between the flat M5-brane and M2-brane

- It is an instance of a class of ‘dyonic’ or ‘dielectric’ solutions, with interpolating supersymmetry

- \( \sin \zeta \) measures the ratio of M5 to M2 brane charges. E.g. \( \sin \zeta = 0 \rightarrow \) the M2-brane charge vanishes
The role of Generalized Calibrations

The set of supersymmetry conditions, constraining the type of $G$-structure, can be nicely reinterpreted in terms of generalized calibrations

- Special holonomy manifolds are characterized by (one or more) covariantly constant form(s) $\Xi$. These forms can be used to characterize special sub-manifolds, which are called 'calibrated'

A 'calibration' is a form $\Xi$, such that:

- $d\Xi = 0$
- $\iota^*_V \Xi \leq \text{vol}_V$ for any tangent plane $V$

A cycle $\Sigma$ is calibrated by $\Xi$ if the inequality is saturated for any plane tangent to $\Sigma$

**Important:** a cycle $\Sigma$ is

- calibrated $\iff$ volume minimizing $\iff$ supersymmetric

E.g. in a $G_2$-holonomy manifold there are the associative ($\phi$) and the co-associative ($\ast \phi$) calibrations

- In the presence of fluxes the notion of calibration should be extended $\rightarrow$ 'generalized calibrations'
The definition is changed replacing the requirement $d\Xi = 0$ with

- $d\Xi = \text{appropriate fluxes}$

'Appropriate' means that the condition can be used to show that a probe brane has \textit{minimal energy} when it wraps a calibrated cycle $\Sigma$:

$$E[\Sigma, \text{flux}] = \text{Mass}[\text{vol}_\Sigma] + WZ[\text{flux}]$$

- The notion of calibration can also be extended in a \textit{different way}, by switching on world-volume fields on a probe brane

- On an M5-brane there is a self-dual three-form $H$. Taking this into account, [Bärwald, Lambert, West] derived a bound for the M5-brane energy $E[\Sigma, H]$

$$\nu + \chi \wedge H \leq \text{vol}_\nu E$$

$\nu, \chi$ are usual bi-linears constructed from parallel spinors

- Using $d\nu = d\chi = 0 = dH \Rightarrow$ A calibrated pair $(H, \Sigma)$, that saturates the bound, gives minimal M5-brane energy in its equivalence class, and is \textit{supersymmetric}
Using the Hamiltonian formulation of the M5-brane, we derived a generalized BPS bound in the presence of background $G$-flux:

$$\nu + \chi \wedge H + C_0 \text{vol}_V \leq \text{vol}_V E$$

- $C_0 \text{vol}_5 = i_k C_6 - \frac{1}{2} i_k C \wedge (C - 2H)$ is a new term
- $k$ is a time-like vector field $\rightarrow k = \frac{\partial}{\partial t}$
- the energy $E$ depends on the background $G$-flux
- $dH = \nu^* G$

A pair $(\Sigma, H)$ is calibrated, i.e. it has minimal M5-brane energy and is supersymmetric iff, the forms $k, \chi, \nu$ satisfy

$$d k = \frac{2}{3} \chi \wedge G - \frac{1}{3} \nu \wedge *G$$
$$d \chi = i_k G$$
$$d \nu = i_k * G - \chi \wedge G$$

- These are the necessary and sufficient conditions for the existence of (at least) one Killing spinor in M-Theory [Gauntlett,Pakis]

- These conditions are equivalent to our set of conditions. E.g. $\chi \sim \sin \zeta dx^1 \wedge dx^2$

$\Rightarrow \sin \zeta$ Measures the amount of M2-brane charge induced on the M5-brane by the WZ coupling with $H$
M-Theory on six-manifolds [Gauntlett, DM, Sparks, Waldram]
(to appear)

Motivations:

- Study the most general (warped) supersymmetric AdS$_5$ solutions in M-Theory $\rightarrow$ dual to 4-dim SCFT

- Characterize 5-dim Minkowski flux compactifications of M-Theory

warped metric $\quad ds_{11}^2 = e^{2\Delta(x)}(ds_5^2 + g_{mn}dx^m dx^n)$

flux $\quad G_{mnpq}$ arbitrary

spinor $\quad \eta = \psi \otimes \xi$

- if $\xi$ is a 6d chiral spinor

$\Rightarrow \quad G = 0$, $\Delta =$constant, $M_6$ is a Calabi-Yau three-fold

(This follows also from the superpotential $W = \int J \wedge G$ [Behrndt, Gukov])

- Taking $\xi$ non-chiral, once again we can have non-trivial flux

- $\xi$ complex, non-chiral spinor $\Leftrightarrow$ SU(2)-structure on $M_6$, characterized by

$$ J, \Omega_{(2,0)}, K^1, K^2 $$
The geometry on $M_6$

- The metric has the canonical form:
  $$d\tilde{s}_6^2 = g_{ij}^4(x, y)dx^i dx^j + \sec^2 \zeta dy^2 + e^{6\Delta} \cos^2 \zeta (d\psi + \rho)^2$$
- $K^2\# = \sec \zeta \frac{\partial}{\partial \psi}$ is a Killing vector $\rightarrow U(1)$ $R$-symmetry
- At any fixed $y$, $M_4$ is Kähler
- $\Omega = e^{i3m\psi} \Omega_0 \Rightarrow \Omega \in \mathcal{L}^k \otimes \Lambda^{2,0}$ ($\mathcal{L}$ is $U(1)$ bundle)
- The flux is completely fixed:
  $$*_6 F = e^{-6\Delta} d(e^{6\Delta} \cos \zeta K^2) - 4m(J - K^1 \wedge K^2 \sin \zeta)$$

The $(2,0)$ form, and the spinors, are charged under the $U(1)$ isometry

- Analogous to the conifold (see [Klebanov, Witten]). The conifold is a cone over the $T^{1,1}$ space:
  $$U(1) \rightarrow T^{1,1} \downarrow \quad S^2 \times S^2$$
  $U(1)$ is an isometry $\rightarrow$ KK reducing gives back $S^2 \times S^2$
  $S^2 \times S^2$ is Kähler, with $(2,0)$-form $\Omega_{(2,0)} = e^{i\varphi} \Omega_0$
  $\varphi \in [0, 4\pi]$ ($k = 2$)
  $\Rightarrow \Omega_{(2,0)}$ has charge 2 $\rightarrow$ spinors have charge 1
Example: $\mathcal{N} = 1$ solution of [Maldacena, Nuñez]

- The solution was constructed in 7d gauged supergravity and uplifted to M-Theory is $\text{AdS}_5 \times_{\text{warped}} M_5$
- It corresponds to the near-horizon limit of M5-branes wrapped on $H^2$

Geometry of $M_6$
- The base is $H^2 \times S^2$ with non-Einstein metric

\[
ds_4^2 = \frac{1}{Y^2} (dX^2 + dY^2) + f(y^2) (d\theta^2 + \sin^2 \theta d\phi^2)
\]
- $\Omega = e^{i3\psi} \quad \psi \in [0, \frac{4\pi}{3}] \quad \Rightarrow \quad \Omega$ has $R$-charge 2
- One can think of $M_6$ as a generalization of the conifold

\[M_6 \text{ Maldacena–Nuñez}\]

$r$ radial direction

$T^{1,1} \text{ base of the cone}$

\[S^1 \to T^{1,1} \quad \downarrow \quad S^2 \times S^2\]

$M_5$ fixed-$y$ sections

\[S^1 \to M_5 \quad \downarrow \quad H^2 \times S^2\]
Supersymmetry

- Generalized Calibrations
- Fluxes
- Intrinsic Torsion of a $G$-Structure
  - AdS/CFT - Flows
  - Compactifications
  - Intersecting Branes
  - Wrapped Branes
Possible directions for future work

- construct new examples of:
  - AdS solutions (AdS/CFT or M-Theory vacua)
  - ‘dielectric solutions’, i.e. solutions with interpolating supersymmetry
  - compactifications to Minkowski space-time (hard, need to include corrections)

- Derive generalized superpotentials for flux compactifications. This is a general problem, i.e. arising also in String Theory compactifications, that can be addressed using generalized calibrations
Generalized calibrations and superpotentials

Special holonomy $\rightarrow$ Calibrations $\rightarrow$ Superpotentials

[Gukov, Vafa, Witten] [Gukov]

$$W = \int (\text{calibration}) \wedge (\text{flux})$$

- M-Theory on $Spin(7)$-manifolds [Acharya, De la Ossa, Gukov]
  [M. Becker, Constantin]

$$W = \int \psi \wedge G$$

$$d\hat{s}_{11}^2 = H^{-2/3}d\hat{s}^2(\mathbb{R}^{1,2}) + H^{1/3}d\hat{s}^2(Spin(7))$$

$$G = vol_3 \wedge d(H^{-1}) + G_{27}$$

$$\tilde{*} \Box H + \frac{1}{2} G_{27} \wedge G_{27} = X_8$$

- We have shown that there exist much more general solutions

  $G$-structure $\rightarrow$ Generalized Calibrations $\rightarrow$ [?] 

- There should be more general superpotentials related to generalized calibrations!

$$W = \int (\text{generalized calibration}) \wedge (\text{flux})$$