IRC Safety

vs.

Calculability

Andrew Larkoski
MIT

AJL, J. Thaler 1307.1699; AJL, I. Moult, D. Neill 1401.4458;
AJL, S. Marzani, G. Soyez, J. Thaler 1402.soon

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Reminder: How to compute differential cross sections in perturbation theory

\[ \frac{d\sigma}{d\mathcal{O}} = \sum_{n} \int d\Pi_{n} |\mathcal{M}_{n}|^{2} \delta (\mathcal{O} - \hat{\mathcal{O}}(\Pi_{n})) \]

\( n = \) number of external particles
\( \mathcal{O} = \) observable

Infrared and Collinear Safety

The phase space constraints imposed by the observable are smooth through real and virtual contributions

\[ \mathcal{M}_{n} = \mathcal{M}^{0}_{n} + \mathcal{M}^{1}_{n} + \mathcal{M}^{2}_{n} + \cdots \]

tree-level \hspace{1cm} one-loop \hspace{1cm} two-loop

Real \hspace{2cm} Virtual

Poster Child: Thrust

\[ \tau \equiv \frac{1}{Q} \sum_{i} E_{i} \sin \theta_{i} \tan \frac{\theta_{i}}{2} \]
Reminder: How to compute differential cross sections in perturbation theory
Reminder: How to compute differential cross sections in perturbation theory

- Fixed-Order Distribution breaks down
  - Logarithms become large
  - Dominant energy flow is along momentum axis

- Must resum logarithms for reliable predictions
  - Soft-Collinear Effective Theory: Resummation by RG evolution
  - Explicit summation of singular approximation to matrix element

- Fixed-Order Distribution is Accurate
  - No large logarithms
  - Dominant emissions are away from soft/collinear region
Reminder: How to compute differential cross sections in perturbation theory

\[
\frac{d\sigma}{d\tau} = (\alpha_s + \cdots) \exp \left[ \sum_n \left( \alpha_s^n \log^{n+1} \tau + \alpha_s^n \log^n \tau + \cdots \right) \right]
\]

- **Leading Logarithms**
  \[\alpha_s \log \tau \sim 1\]
  \[\tau \sim 1\]

- **Next-to-Leading Logarithms**
  \[\frac{d\sigma}{d\tau} = \alpha_s + \alpha_s^2 + \alpha_s^3 + \cdots\]
Reminder: How to compute differential cross sections in perturbation theory

\[
\frac{d\sigma}{d\tau} = (\alpha_s + \cdots) \exp \left[ \sum_n (\alpha_s^n \log^{n+1} \tau + \alpha_s^n \log^n \tau + \cdots) \right]
\]

- **Leading Logarithms**
  - \(\alpha_s \log \tau \sim 1\)
  - \(\tau \sim 1\)

- **Next-to-Leading Logarithms**
  - \(\frac{d\sigma}{d\tau} = \alpha_s + \alpha_s^2 + \alpha_s^3 + \cdots\)

Match distributions in different phase space regions
Reminder: How to compute differential cross sections in perturbation theory

\[
\frac{d\sigma}{d\tau} = (\alpha_s + \cdots) \exp \left[ \sum_n (\alpha_s^n \log^{n+1} \tau + \alpha_s^n \log^n \tau + \cdots) \right]
\]

Leading Logarithms

Next-to-Leading Logarithms

Match distributions in different phase space regions

\[
\frac{d\sigma}{d\tau} = \alpha_s + \alpha_s^2 + \alpha_s^3 + \cdots
\]

\[
\tau \sim 1
\]

\[
\alpha_s \log \tau \sim 1
\]

\[
\tau \gg \frac{\Lambda_{QCD}}{Q}
\]

Non-perturbative effects dominate

Non-perturbative effects are power corrections (OPE)
Ratio Observables in Perturbation Theory

A, B: IRC safe observables

Real contribution: divergent for all $r$

Virtual contribution: divergent, proportional to $\delta(r)$

Singular region of phase space
A, B: IRC safe observables

Real contribution: divergent for all $r$

Virtual contribution: divergent, proportional to $\delta(r)$

**IRC Unsafe!?**

Soyez, Salam, Kim, Dutta, Cacciari 2012

Standard computation methods are useless for these observables

A, B can be measured separately; why can’t their ratio?

This is a major practical issue
Example: $N$-subjettiness  

$$
\tau_N^{(\beta)} = \sum_{i \in J} p_{Ti} \min\{R_{i1}^{\beta}, R_{i2}^{\beta}, \ldots, R_{iN}^{\beta}\}
$$

\[ \tau_2^{(\beta)} = \frac{\tau_2^{(\beta)}}{\tau_1^{(\beta)}} \] Powerful boosted $W$ tagger  
Selects for 2-subjet structures

\[ \tau_3^{(\beta)} = \frac{\tau_3^{(\beta)}}{\tau_2^{(\beta)}} \] Powerful boosted $t$ tagger  
Selects for 3-subjet structures

Other ratio observables used for jet substructure analysis:

- Energy correlation functions  
  AJL, Salam, Thaler 2013
- Angular correlation functions  
  Jankowiak, AJL 2011
- Planar flow  
  Almeida, Lee, Perez, et al. 2008

Why does Monte Carlo model data so well?
Ratio Observables in Perturbation Theory

Assume $A \leq B$

$0 \leq r \leq 1$

Need to regulate the singular region of phase space for calculability
Ratio Observables in Perturbation Theory

Assume $A \leq B$

$0 \leq r \leq 1$

Need to regulate the singular region of phase space for calculability

1) Explicit cut on denominator observable $B$

May introduce undesired logarithmic sensitivity to $B_{\text{cut}}$
Ratio Observables in Perturbation Theory

Assume $A \leq B$

$0 \leq r \leq 1$

Need to regulate the singular region of phase space for calculability

1) Explicit cut on denominator observable $B$

May introduce undesired logarithmic sensitivity to $B_{\text{cut}}$

2) Include emissions to all-orders in perturbation theory

Exponentially suppresses singular region organically
Ratio Observables in Perturbation Theory

Definition: \( \frac{d\sigma}{dr} = \int dA dB \frac{d^2\sigma}{dA dB} \delta \left( r - \frac{A}{B} \right) \)

\( \frac{d^2\sigma}{dA dB} \) is the fundamental object

Well-defined order-by-order in perturbation theory

Follows from IRC safety of A and B

To all-orders, singular region is exponentially suppressed by perturbative Sudakov factor

Marginalization is well-defined

Ratio observable is “Sudakov safe”
Example: Ratio of Angularities

- IRC Unsafety at fixed order
- Sudakov Safety at all-orders
- Controlled non-perturbative sensitivity

Looking Forward

- Higher-order effects
- Other examples of Sudakov Safe observables

Conclusions
Ex: Ratio of Angularities
Angularities Measured on Jets

Recoil-free jet axis
(broadening, winner-take-all)

AJL, Neill, Thaler 2014

Want to measure:

\[
\frac{d\sigma}{dr} = \int d\epsilon_\alpha d\epsilon_\beta \frac{d^2\sigma}{d\epsilon_\alpha d\epsilon_\beta} \delta \left( r - \frac{\epsilon_\alpha}{\epsilon_\beta} \right)
\]

We take \( \alpha > \beta \) so \( \epsilon_\alpha < \epsilon_\beta \)

\[ 0 \leq r \leq 1 \]

Compute double differential cross section to different accuracies
Double Differential Cross Section:

Energy conservation: \( z < 1 \) \( \rightarrow \) \( e_\beta < e_\alpha \)

Emission within jet: \( \theta < R_0 \) \( \rightarrow \) \( e_\alpha < e_\beta \)

Singular matrix element:

\[
S(z, \theta) dz \, d\theta = 2 \frac{\alpha_s}{\pi} C_F \frac{d\theta}{\theta} \frac{dz}{z}
\]

Double Differential Cross Section:

\[
\frac{1}{\sigma} \frac{d^2 \sigma}{de_\alpha \, de_\beta} = 2 \frac{\alpha_s}{\pi} C_F \frac{1}{\alpha - \beta} \frac{e_\alpha - e_\beta}{e_\alpha \, e_\beta} \Theta(e_\beta - e_\alpha) \Theta(e_\beta - e_\alpha)
\]
Fixed-Order Distribution

\[
\frac{1}{\sigma} \frac{d^2 \sigma}{d e_\alpha \, d e_\beta} = 2 \frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \frac{1}{e_\alpha e_\beta} \Theta (e_\beta - e_\alpha) \Theta (e_\beta - e_\alpha)
\]

\[ r = \frac{e_\alpha}{e_\beta} \]

\[ e_\beta^{e_\beta} < e_\alpha^{e_\alpha} \rightarrow e_\beta < r^{\frac{\beta}{\alpha - \beta}} \]

Measuring \( r \) does not regulate \( e_\beta \) singularity!

\[ e_\alpha < e_\beta \rightarrow r < 1 \]

\[
\frac{1}{\sigma} \frac{d \sigma}{d r} = 2 \frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \frac{1}{r^{\frac{\beta}{\alpha - \beta}}} \int_0^r \frac{d e_\beta}{e_\beta}
\]

Logarithmic sensitivity to any lower bound

Ratio observable cross section undefined at fixed order:
IRC unsafe
Double-Logarithmic Distribution

Probability for an emission:

\[ 2 \frac{\alpha_s}{\pi} C_F d\theta \frac{dz}{z} = 2 \frac{\alpha_s}{\pi} C_F d\log \frac{1}{\theta} d\log \frac{1}{z} \]

Emissions are uniformly distributed in the plane.
Double-Logarithmic Distribution

Probability for an emission:
\[ 2 \frac{\alpha_s}{\pi} C_F \frac{d\theta}{\theta} \frac{dz}{z} = 2 \frac{\alpha_s}{\pi} C_F \, d\log \frac{1}{\theta} \, d\log \frac{1}{z} \]

Emissions are uniformly distributed in the plane

\[ \log \frac{1}{e_\alpha} = \log \frac{1}{z} + \alpha \log \frac{R_0}{\theta} \]

\[ P(x < e_\alpha) = e^{-2 \frac{\alpha_s}{\pi} C_F} \]

\[ \Sigma(e_\alpha) = e^{-\frac{\alpha_s}{\pi} \frac{C_F}{\alpha} \log^2 e_\alpha} \]

\[ \frac{1}{\sigma} \frac{d\sigma}{de_\alpha} = \frac{\partial}{\partial e_\alpha} \Sigma(e_\alpha) \]
Double-Logarithmic Distribution

Probability for an emission:

\[ 2 \frac{\alpha_s}{\pi} C_F \frac{d\theta}{\theta} \frac{dz}{z} = 2 \frac{\alpha_s}{\pi} C_F d \log \frac{1}{\theta} d \log \frac{1}{z} \]

\[ \log \frac{1}{e_\alpha} = \log \frac{1}{z} + \alpha \log \frac{R_0}{\theta} \]

\[ \log \frac{1}{e_\beta} = \log \frac{1}{z} + \beta \log \frac{R_0}{\theta} \]

\[ P(x < e_\alpha, y < e_\beta) = e^{-2 \frac{\alpha_s}{\pi} C_F} \]

\[ \Sigma(e_\alpha, e_\beta) = e \]

\[ \frac{1}{\sigma} \frac{d^2 \sigma}{de_\alpha de_\beta} = \frac{\partial^2}{\partial e_\alpha \partial e_\beta} \Sigma(e_\alpha, e_\beta) \]
Double-Logarithmic Distribution

\[ \Sigma(e_\alpha, e_\beta) = e \]

(Phase space constraints suppressed)

\[ \frac{1}{\sigma} \frac{d\sigma}{dr} = \int de_\alpha de_\beta \frac{\partial^2}{\partial e_\alpha \partial e_\beta} \Sigma(e_\alpha, e_\beta) \delta \left( r - \frac{e_\alpha}{e_\beta} \right) \]

\[ = \frac{\sqrt{\alpha_s C_F \beta} 1}{\alpha - \beta} \frac{1}{r} \left( 1 - 2 \frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \log^2 r \right) \left( \text{erf} \left[ \frac{\sqrt{\alpha_s C_F \beta}}{\sqrt{\pi(\alpha - \beta)} \log r} \right] + 1 \right) e^{-\frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \log^2 r} \]

\[ - 2 \frac{\alpha_s}{\pi} \frac{C_F}{\alpha - \beta} \frac{\log r}{r} e^{-\frac{\alpha_s}{\pi} \frac{C_F}{(\alpha - \beta)^2} \log^2 r} \]

Expanding in \( \alpha_s \):

\[ \frac{1}{\sigma} \frac{d\sigma}{dr} = \sqrt{\alpha_s} \frac{\sqrt{C_F \beta} 1}{\alpha - \beta} \frac{1}{r} + O \left( (\sqrt{\alpha_s})^2 \right) \]

No Taylor expansion about \( \alpha_s = 0! \)
Double-Logarithmic Distribution

\[ \frac{1}{\sigma} \frac{d\sigma}{dr} = \sqrt{\alpha_s} \sqrt{C_F} \beta \frac{1}{\alpha - \beta} r + \mathcal{O}\left( (\sqrt{\alpha_s})^2 \right) \]

Consequences:

IRC unsafe ↔ No Taylor series about \( \alpha_s = 0 \)

“Sudakov safe”: finite cross section with all-orders included

Observations:

Taylor series expansion about \( \alpha_s \neq 0 \)

Can this cross section be computed with CFT techniques?

Connections:

Anomalous dimension of fragmentation function moments

\( j \neq 1: \gamma(j, \alpha_s) = \frac{\alpha_s C_A}{\pi} \frac{1}{j - 1} + \mathcal{O}(\alpha_s^2) \)

\( j = 1: \gamma(j = 1, \alpha_s) = \sqrt{\frac{\alpha_s C_A}{2\pi}} \)
Practical Consequences of Calculability: Monte Carlos and the ratio observable

Monte Carlos approximate all-orders exclusive cross sections

Should accurately reproduce the cross section for the ratio near \( r = 0 \)

Deviation for \( r \sim 1 \) where multiple emissions become important

Monte Carlo Resummation
Ratio \( r_{\alpha,\beta}, \alpha = 1 \)
- \( \beta = 0.75: \) LL, LL+MC
- \( \beta = 0.5: \) LL, LL+MC
- \( \beta = 0.25: \) LL, LL+MC

Monte Carlo Resummation
Ratio \( r_{\alpha,\beta}, \alpha = 2 \)
- \( \beta = 1.5: \) LL, LL+MC
- \( \beta = 1.0: \) LL, LL+MC
- \( \beta = 0.5: \) LL, LL+MC
Practical Consequences of Calculability: Non-perturbative Corrections

Standard lore: IRC safe observables have controlled sensitivity to non-perturbative physics

Ex: Thrust

\[
\tau \equiv \frac{1}{Q} \sum_i E_i \sin \theta_i \tan \frac{\theta_i}{2}
\]

OPE region:

\[
\frac{d\sigma}{d\tau} = \frac{d\sigma_{\text{pert}}}{d\tau} + O\left(\frac{\Lambda_{QCD}}{Q}\right)
\]

\[
\tau \sim \frac{\Lambda_{QCD}}{Q}
\]

\[
\tau \gg \frac{\Lambda_{QCD}}{Q}
\]

Dominated by non-perturbative physics
Practical Consequences of Calculability: Non-perturbative Corrections

Assume:
\[ \frac{d^2 \sigma}{de_\alpha de_\beta} = \frac{d^2 \sigma^{\text{pert}}}{de_\alpha de_\beta} + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}}{Q} \right) \]

\[ e_\alpha, e_\beta \gg \frac{\Lambda_{\text{QCD}}}{Q} \]
\[ \Lambda_{\text{QCD}} \ll \eta \ll Q \]

\[ \frac{d\sigma}{dr} = \int_{\eta/Q} \frac{d\sigma^{\text{pert}}}{de_\alpha de_\beta} \frac{d^2 \sigma}{de_\alpha de_\beta} + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}}{Q} \right) \delta \left( r - \frac{e_\alpha}{e_\beta} \right) \]
\[ + \int_{\eta/Q} \frac{d\sigma}{de_\alpha de_\beta} \delta \left( r - \frac{e_\alpha}{e_\beta} \right) \]

NP effects are power corrections

Dominated by NP effects

Direct non-perturbative contribution is small for:
\[ Q \gg \eta e^{2\sqrt{\alpha_s} \sqrt{C_F}} \equiv Q_{\text{Sud}} \]

Requires finite \( \alpha_s \)

NP effects are power-suppressed at large \( Q \) for \( r \sim 1 \)

For reasonable values of the parameters, \( Q_{\text{Sud}} \sim 250 \text{ GeV} \)
Practical Consequences of Calculability: Non-perturbative Corrections

Power-suppressed non-perturbative corrections!
Review

Ratio observables are IRC unsafe but calculable: “Sudakov safe”

Cross section series in $\alpha_s^{1/2}$

Monte Carlo should describe these observables accurately

Includes all-orders approximation to matrix element

Non-perturbative corrections are power-suppressed

Similar behavior as with IRC safe observables
Going Further

Calculating double differential cross section to higher accuracy

Fixed-Order Corrections

Schematic form:

\[
\frac{d^2\sigma^\text{match}}{de_\alpha de_\beta} = \frac{d^2\sigma^\text{resum}}{de_\alpha de_\beta} + \frac{d^2\sigma^\text{FO}}{de_\alpha de_\beta} - \frac{d^2\sigma^\text{sing}}{de_\alpha de_\beta}
\]

Bulk contribution to ratio cross section

Small (power-suppressed) corrections

Log–R Matching
Ratio \( r_{\alpha,\beta}, \alpha = 2 \)

\[ \beta = 1.5: \quad \text{LL} \quad \text{LL+LO} \]
\[ \beta = 1.0: \quad \text{LL} \quad \text{LL+LO} \]
\[ \beta = 0.5: \quad \text{LL} \quad \text{LL+LO} \]

Log–R Matching
Ratio \( r_{\alpha,\beta}, \alpha = 1 \)

\[ \beta = 0.75: \quad \text{LL} \quad \text{LL+LO} \]
\[ \beta = 0.5: \quad \text{LL} \quad \text{LL+LO} \]
\[ \beta = 0.25: \quad \text{LL} \quad \text{LL+LO} \]
Going Further

Calculating double differential cross section to higher accuracy

Higher-order Resummation

AL, Moult, Neill 2014

Hard

Jet

Soft

\[ H(\mu) \times J(\tau, \mu) \otimes S(\tau, \mu) \]
Going Further

Calculating double differential cross section to higher accuracy

Higher-order Resummation

Constraints:

\[ \Sigma(e_\alpha, e_\beta = e_\beta^{\alpha/\beta}) = \Sigma(e_\alpha) \]
\[ \Sigma(e_\alpha = e_\beta, e_\beta) = \Sigma(e_\beta) \]
\[ \frac{\partial}{\partial e_\alpha} \Sigma(e_\alpha, e_\beta) \bigg|_{e_\beta = e_\beta^{\alpha/\beta}} = \frac{d\sigma}{de_\alpha} \]
\[ \frac{\partial}{\partial e_\beta} \Sigma(e_\alpha, e_\beta) \bigg|_{e_\alpha = e_\beta} = \frac{d\sigma}{de_\beta} \]

\[ \sum(e_\alpha, e_\beta) \]

Defined by interpolating function that satisfies boundary conditions!

Unique up to \( O(\alpha_s^4) \)

“Holographic Factorization Theorem”

\[ \frac{d\sigma}{d\tau} \simeq H(\mu) \times J(\tau, \mu) \otimes S(\tau, \mu) \]
Going Further

Other Examples of Sudakov Safety: Groomed Energy Loss

Original jet: $E_0$

Soft Drop
Grooming

Groomed jet: $E_g$

\[ \Delta_E \equiv \frac{E_0 - E_g}{E_0} \]

\[ \Sigma_{\text{energy-drop}}(\Delta_E) = \frac{\log z_{\text{cut}} - B_i}{\log \Delta_E - B_i} + \frac{\pi \beta}{2C_i \alpha_s (\log \Delta_E - B_i)^2} \left( 1 - e^{-2 \frac{\alpha_s}{\pi} \frac{C_i}{\beta} \log \frac{z_{\text{cut}}}{\Delta_E} \left( \log \frac{1}{\Delta_E} + B_i \right)} \right) \]

\[ \alpha_s \text{ expansion: } \Sigma_{\text{energy-drop}}(\Delta_E) = 1 - \frac{\alpha_s}{\pi} \frac{C_i}{\beta} \log^2 \frac{z_{\text{cut}}}{\Delta_E} + \mathcal{O} \left( \left( \frac{\alpha_s}{\beta} \right)^2 \right) \]

\[ \beta = 0: \quad \Sigma_{\text{energy-drop}}(\Delta_E)_{\beta=0} = \frac{\log z_{\text{cut}} - B_i}{\log \Delta_E - B_i} \quad \text{independent of } \alpha_s! \]
Conclusions

What other examples of Sudakov safe observables are there?

Can these techniques be applied to observables like \(N\)-subjettiness?

Do we need a new definition of IRC safety/calculability in perturbation theory?

Can techniques from CFTs help in understanding these observables?

Is QCD better approximated by a free theory, a weakly coupled CFT or something else?