Perturbing the $U(1)$ Dirac Spin Liquid State in Spin-1/2 kagome
Raman scattering, magnetic field, and hole doping

Wing-Ho Ko
MIT
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Acknowledgments

Xiao-Gang Wen

Patrick Lee

Tai-Kai Ng

Ying Ran

Zheng-Xin Liu
Outline

1. The Spin-1/2 Kagome Lattice
2. Derivation and Properties of the $U(1)$ DSL State
3. Raman Scattering
4. External Magnetic Field
5. Hole Doping

Reference: Ran, Ko, Lee, & Wen, PRL 102, 047205 (2009)
Ko, Lee, & Wen, PRB 79, 214502 (2009)
Ko, Liu, Ng, & Lee, PRB 81, 024414 (2010)
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The nearest-neighbor antiferromagnetic Heisenberg model is highly frustrated on the kagome lattice.

• kagome lattice = 2D lattice of corner-sharing triangles.

• Simple Néel order does not work well on the kagome lattice.

• Classical ground states:
  \[ \sum_{i \in \Delta} \mathbf{S}_i = 0 \, . \]
  • \# classical ground states \( \propto e^N \).

[Chalker et al., PRL 68, 855 (1992)].
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In the quantum case, singlet formation is possible and may be favored.

- 1D chain:
  
  \[ E_{\text{Néel}} = S^2 J \]  
  \[ E_{\text{singlet}} = \frac{1}{2} S(S + 1)J \]  
  per bond

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- kagome is highly frustrated \(\Rightarrow\) rare opportunity of realizing a singlet ground state.
Two major classes of singlet states: valence bond solids (VBS) and spin liquid (SL).

- In a VBS, certain singlet bonds are preferred, which results in a symmetry-broken state.

- In a SL, different bond configurations superpose, which results in a state that breaks no lattice symmetry.

- A VBS state generally has a spin gap, while a SL state can be gapped or gapless.
For $S=1/2$ kagome, the leading proposals are the 36-site VBS and the U(1) Dirac spin liquid.

- The 36-site VBS pattern is found in series expansion [Singh and Huse, PRB 76, 180407 (2007)] and entanglement renormalization [Evenbly and Vidal, arXiv:0904.3383].

- From VMC, the $U(1)$ Dirac spin liquid (DSL) state has the lowest energy among various SL states, and is stable against small VBS perturbations [e.g., Ran et al., PRL 98, 117205 (2007)].

- Exact diagonalization: initially found small ($\sim J/20$) spin gap; now leaning towards a gapless proposal [Waldtmann et al., EPJB 2, 501 (1998); arXiv:0907.4164].
Experimental realization of $S = 1/2$ kagome: Herbertsmithite $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$.

- Herbertsmithite: layered structure with Cu forming an AF kagome lattice.
- Caveats: Zn impurities and Dzyaloshinskii–Moriya interactions.
- Experimental Results [e.g., Helton et al., PRL 98, 107204 (2007); Bert et al., JP:CS 145, 012004 (2009)]:
  - Neutron scattering: no magnetic order down to 1.8 K.
  - $\mu$SR: no spin freezing down to 50 $\mu$K.
  - Heat capacity: vanishes as a power law as $T \rightarrow 0$.
  - Spin susceptibility: diverges as $T \rightarrow 0$.
  - NMR shift: power law as $T \rightarrow 0$. 
Research motivation: Deriving further experimental consequences of the $U(1)$ DSL state.

- All experiments point to a state without magnetic order. But more data is needed to tell if it is a VBS state or a SL state, and which VBS/SL state it is.
- Without concrete theory, the experimental data are hard to interpret.
- Without concrete theory, unbiased theoretical calculations are difficult.
- The $U(1)$ Dirac spin-liquid state is a theoretically interesting exotic state of matter.

Thus, our approach: Assume the DSL state and consider further experimental consequences.
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Deriving the $U(1)$ DSL state: Slave boson formulation

- Start with Heisenberg (or more generally $t-J$) model:
  $$H_{tJ} = \sum_{\langle ij \rangle} J (S_i \cdot S_j - \frac{1}{4} n_i n_j) - t \left( c_{i\sigma}^\dagger c_{j\sigma} + h.c. \right); \quad \sum_\sigma c_i^\dagger c_i \leq 1$$

- Apply the slave boson decomposition [Lee et al., RMP 78, 17 (2006)]:
  $$S_i = \frac{1}{2} \sum_{\alpha,\beta} f_i^{\dagger \alpha} \tau_{\alpha,\beta} f_i^\beta; \quad c_i^\dagger = f_i^\dagger h_i; \quad f_i^\dagger f_i^\dagger + f_i^\dagger f_i^\dagger + h_i^\dagger h_i = 1$$

- Decouple four-operator terms by a Hubbard–Stratonovich transformation, with the following ansatz:
  $$\chi_{ij} \equiv \sum_\sigma \langle f_{i\sigma}^\dagger f_{j\sigma} \rangle = \chi e^{i\alpha_{ij}}; \quad \Delta_{ij} \equiv \langle f_{i\uparrow}^\dagger f_{j\downarrow} - f_{i\downarrow}^\dagger f_{j\uparrow} \rangle = 0$$

- This results in a mean-field Hamiltonian....
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  \]
  \[\text{spinon} \quad \text{holon} \quad \text{Constraint enforced by Lagrange multiplier } \alpha_i^0\]

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Deriving the $U(1)$ DSL state: Emergent gauge field

$$H_{\text{MF}} = \sum_{i\sigma} f_{i\sigma}^\dagger (i\alpha_0^i - \mu_F) f_{i\sigma} - \frac{3\chi J}{8} \sum_{\langle ij \rangle,\sigma} (e^{i\alpha_{ij}} f_{i\sigma}^\dagger f_{j\sigma} + \text{h.c.})$$

$$+ \sum_i h_i^\dagger (i\alpha_0^i - \mu_B) h_i - t\chi \sum_{\langle ij \rangle} (e^{i\alpha_{ij}} h_i^\dagger h_j + \text{h.c.})$$

- The $\alpha$ field is an emergent gauge field, corresponding to gauge symmetry $f_i^\dagger \mapsto e^{i\theta} f_i^\dagger, h \mapsto e^{-i\theta} h$.

- At the lattice level $\alpha$ is a compact gauge field (i.e., monopoles are allowed).

- But with Dirac fermions, the system can be in a deconfined phase (i.e., monopoles can be neglected) [Hermele et al., PRB 70, 214437 (2004)].
**Deriving the \( U(1) \) DSL state: Band Structure**

\[
H_{MF} = \sum_{i\sigma} f_{i\sigma}^\dagger (i\alpha_i^0 - \mu_F) f_{i\sigma} - \frac{3\chi J}{8} \sum_{<ij>,\sigma} (e^{i\alpha_{ij}} f_{i\sigma}^\dagger f_{j\sigma} + h.c.) \\
+ \sum_{i} h_{i}^\dagger (i\alpha_i^0 - \mu_B) h_i - t\chi \sum_{<ij>} (e^{i\alpha_{ij}} h_{i}^\dagger h_{j} + h.c.)
\]

- Neglecting fluctuation of \( \alpha \), spinons and holons are decoupled.
- Mean-field ansatz for SL state can be specified by pattern of \( \alpha \) flux.
- \( U(1) \) Dirac spin liquid state: \( \pi \) flux per \( \Box \) and 0 flux per \( \triangle \).
- \( \pi \) flux \( \implies \) unit cell doubled in band structures.
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Properties of the $U(1)$ DSL state: Band structure

Real space

$k$-space

$t_{\text{eff}} = +t$ ; $t_{\text{eff}} = -t$
Properties of the $U(1)$ DSL state: Band structure

**Real space**

$\begin{align*}
\text{positive } t_{\text{eff}} &= +t \\
\text{negative } t_{\text{eff}} &= -t
\end{align*}$

**k-space**

- $-2\pi \sqrt{3}$
- $-\pi \sqrt{3}$
- $0$
- $\pi \sqrt{3}$
- $2\pi \sqrt{3}$
Properties of the $U(1)$ DSL state: Band structure

Real space

\[ t_{\text{eff}} = +t \; ; \quad t_{\text{eff}} = -t \]

$k$-space
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Real space

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$k$-space

\[ k_x \quad k_y \]

\[ x^2 \]

\[ -2\pi/3 \quad -\pi/3 \quad \pi/3 \quad 2\pi/3 \]

\[ -\pi' \quad -\pi/2 \quad \pi/2 \quad \pi' \]

\[ k_x \]

\[ k_y \]
Properties of the $U(1)$ DSL state: Band structure

Real space

\[ t_{\text{eff}} = +t; \quad t_{\text{eff}} = -t \]

$k$-space

\[ x^2 - \frac{2\pi}{\sqrt{3}} k_y \]

\[ x^2 - \frac{\pi}{2} k_x \]

\[ x^2 - \pi k_x \]

\[ x^2 - \pi' k_x \]

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Properties of the $U(1)$ DSL state: Band structure

Real space

$k$-space

$t_{\text{eff}} = +t$; $t_{\text{eff}} = -t$
Properties of the $U(1)$ DSL state: Thermodynamics and correlation

- At low energy, the $U(1)$ DSL state is described by QED$_3$.
  - i.e., gauge field coupled to Dirac fermions in (2+1)-D.
- Thermodynamics of the $U(1)$ DSL state is dominated by the spinon Fermi surface.
  - Zero-field spin susceptibility: $\chi(T) \sim T$.
  - Heat capacity: $C_V(T) \sim T^2$.
- $U(1)$ DSL state is “quantum critical” — many correlations decay algebraically [Hermele et al., PRB 77, 224413 (2008)].
  - Emergent SU(4) symmetry among Dirac nodes $\implies$ different correlations can have the same scaling dimension.
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Raman scattering in Mott-Hubbard system: the Shastry–Shraiman formulation

- Raman scattering = inelastic scattering of photon.
  - Good for studying excitations of the system.
  - Probe only excitations with $q \approx 0$.
- We are concerned with a half-filled Hubbard system, in the regime where $|\omega_i - \omega_f| \ll U$ and $\omega_i \approx U$.
  - Both initial and final states are spin states.
  - $\implies$ T-matrix can be written in terms of spin operators.
- Intermediate states are dominated by the sector where $\sum_i n_{i\uparrow} n_{i\downarrow} = 1$.
  - The T-matrix can be organized as an expansion in $t/(U - \omega_i)$

$T^{(2)} \sim \frac{t^2}{U - \omega_i} S_i \cdot S_j + \ldots$

[Shastry & Shraiman, IJMPB 5, 365 (1991)].
Spin-chirality terms in the Shastry–Shraiman formulation

- Because of holon-doublon symmetry, there is no $t^3$ order contribution.

- For the square and triangular lattice, there is no $t^4$ order contribution to spin-chirality because of a non-trivial cancellation between 3-site and 4-site pathways.

- But such cancellation is absent in the kagome lattice.
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\[ i_1 + i_2,3 = 0 \]

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$$
\begin{align*}
\text{i} & \quad \text{iii} \\
\text{ii} & \quad \text{iii} \\
\text{i} & \quad \text{ii}
\end{align*}
+ 
\begin{align*}
\text{i} & \quad \text{iii} \\
\text{ii} & \quad \text{iii} \\
\text{i} & \quad \text{ii}
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\begin{align*}
\text{i} & \quad \text{iii} \\
\text{ii} & \quad \text{iii} \\
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\end{align*} + 
\begin{align*}
\text{iv} & \quad \text{iii} \\
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\begin{align*}
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Raman T-matrix for the kagome geometry

- For kagome geometry, the Raman T-matrix decomposes into 3 irreps:

\[ T = T(A_{1g})(\bar{x}x + \bar{y}y) + T(E_g^{(1)})(\bar{x}x - \bar{y}y) + T(E_g^{(2)})(\bar{y}y + \bar{x}x) + T(A_{2g})(\bar{x}y - \bar{y}x) \]

- To lowest order in inelastic terms,

\[
T(E_g^{(1)}) = \frac{4t^2}{\omega_i - U} \left( \frac{1}{4} \left( \sum_{\langle ij \rangle} + \sum_{\langle ij \rangle} - 2 \sum_{\langle ij \rangle} \right) S_i \cdot S_j \right)
\]

\[
T(E_g^{(2)}) = \frac{4t^2}{\omega_i - U} \sqrt{3} \left( \frac{1}{4} \left( \sum_{\langle ij \rangle} - \sum_{\langle ij \rangle} \right) S_i \cdot S_j \right)
\]

\[
T(A_{1g}) = \frac{-t^4}{(\omega_i - U)^3} \left( 2 \sum_{\langle\langle ij \rangle\rangle} + \sum_{\langle ij \rangle} \right) S_i \cdot S_j
\]

\[
T(A_{2g}) = \frac{2\sqrt{3}it^4}{(\omega_i - U)^3} \sum R \left( 3 \triangle + 3 \triangle' + \langle \right) + \langle \right.
\]

\[
+ \langle + \langle \right. + \langle \right. + \langle \right. + \langle \right. + \langle \right.
\]

\[
( \langle i \rightarrow j \leftarrow k \rangle = S_i \cdot S_j \times S_k, \text{ etc.} )
\]
Raman signals: spinon-antispinon pairs and gauge mode

- \( \mathbf{S}_i \cdot \mathbf{S}_j \sim f^\dagger f f^\dagger f \) and \( \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) \sim f^\dagger f f^\dagger f f^\dagger f \)
  \( \Rightarrow \) contributions from spinon-antispinon pairs.
- \( \Rightarrow \) continuum of signal \( I_\alpha (\Delta \omega) = |\langle f | O_\alpha | i \rangle|^2 \text{DOS}(\Delta \omega) \).

- At low energy, one-pair states dominates.
- For Dirac node, \( \text{DOS}_{1\text{pair}} \sim \mathcal{E} \), and matrix element is suppressed in \( E_g \) and \( A_{1g} \), but not in \( A_{2g} \).
  \( \Rightarrow \) Spinon-antispinon pairs contribute \( I_{A_{2g}} (\Delta \omega) \sim \mathcal{E} \) and \( I_{E_g/A_{1g}} (\Delta \omega) \sim \mathcal{E}^3 \) at low energy.

- However, an additional collective excitation is available in \( A_{2g} \):
  \( \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k \sim i \chi^3 \exp(i \oint_{\triangle} \alpha \cdot d\ell) + h.c. \sim \chi^3 \iint_{\triangle} b d^2x \)
  \( \Rightarrow \) \( I_{A_{2g}} \sim \langle \Phi_b \Phi_b \rangle + \ldots \sim \chi^2 \langle \alpha \alpha \rangle + \ldots \)
  (Recall that \( \langle f_i^\dagger f_j \rangle \sim \chi \exp(i \alpha_{ij}) \) )
- In our case (QED\(_3\) with Dirac fermions), turns out that \( \langle \alpha \alpha \rangle \sim 1/\omega \) when \( q \approx 0 \) [Ioffe & Larkin, PRB 39, 8988 (1989)].

- Analogy: plasmon mode vs. particle-hole continuum in normal metal.
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Raman signals: spinon-antispinon pairs and gauge mode

\[ A_{1g}, E_g \]

Intensity

\[ \propto \omega^3 \]

Energy shift

\[ A_{2g} \]

Intensity

\[ \propto \omega \]

\[ \propto \frac{1}{\omega} \]

Energy shift
Experimental Comparisons: Some qualitative agreements

- Wulferding and Lemmens [unpublished] have obtained Raman signal on herbertsmithite.
  - At low T, data shows a broad background with a near-linear piece at low-energy.
  - Roughly agree with the theoretical picture presented previously.
Outline

1. The Spin-1/2 Kagome Lattice
2. Derivation and Properties of the $U(1)$ DSL State
3. Raman Scattering
4. External Magnetic Field
5. Hole Doping
External Magnetic Field

External magnetic field and the formation of Landau levels

- In Mott systems, B-field causes only Zeeman splitting.
  - This induces spinon and antispinon pockets near the Dirac node.
- However, with the emergent gauge field $\alpha$, Landau levels can form spontaneously.
- From VMC calculations,
  \[
  \Delta e_{FP}^{Prj} \approx 0.33(2)B^{3/2} + 0.00(4)B^2 \\
  \Delta e_{LL}^{Prj} \approx 0.223(6)B^{3/2} + 0.03(1)B^2 \\
  \Delta e_{MF}^{LL} < \Delta e_{MF}^{FP} \text{ to leading order in } \Delta n \\
  \implies \text{Landau level state is favored.}
  \]
$S_z$ density fluctuation as gapless mode

- $b$ is an emergent gauge field $\Rightarrow$ its strength can fluctuate in space.
- The fluctuation of $b$ is tied to the fluctuation of $S_z$ density.
- In long-wavelength limit, energy cost of $b$ fluctuation $\rightarrow 0$ $\Rightarrow$ The system has a gapless mode!
  - Derivative expansion $\Rightarrow$ linear dispersion.
- Other density fluctuations and quasiparticle excitations are gapped $\Rightarrow$ gapless mode is unique.
- Mathematical description given by Chern–Simons theory.
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Gapless mode and XY-ordering

- Recap: we found a single linearly-dispersing gapless mode, which looks like...
- A Goldstone boson! And indeed it is.
  - Corresponding to this Goldstone mode is a spontaneously broken XY order.
- Analogy:
  - Superfluid: \( \hat{\psi} = \sqrt{\hat{\rho}} e^{-i\hat{\theta}} \), \([\hat{\rho}, \hat{\theta}] = i\), gapless \( \rho \) fluctuation \( \xrightarrow{} \) ordered (SF) phase;
  - XY model: \( \hat{S}^+ = e^{i\hat{\theta}} \), \([\hat{S}_z, \hat{\theta}] = i\), gapless \( S_z \) fluctuation \( \xrightarrow{} \) XY ordered phase.
- VMC found the \( q = 0 \) order.
- \( S^+ \) in XY model \( \sim \) \( V^\dagger \) in QED\(_3\)
  \( \xrightarrow{} \) in-plane magnetization \( M \sim B^\gamma \).
  (\( V^\dagger \) monopole operator, \( \gamma \) its scaling dimension)
Outline

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Recap: Band structure

Real space

$$t_{\text{eff}} = +t; \quad t_{\text{eff}} = -t$$

k-space

Perturbing the U(1) DSL State
Hole doping and formation of Landau levels

- Doping can in principle be achieved by substituting Cl with S.
- In slave-boson picture, hole doping introduces holons and antispinons.
- As before, an emergent $b$ field can open up Landau levels in the spinon and holon bands.
  - At mean-field, $\Delta E_{\text{spinon}} \sim -B^{3/2}$ while $\Delta E_{\text{holon}} \sim B^2$
    $\Rightarrow$ LL state favored.
  - At mean-field, $b$ optimal when antispinons form $\nu = -1$ LL state
    $\Rightarrow$ holons form $\nu = 1/2$ Laughlin state.
Charge fluctuation, Goldstone mode, and superconductivity

- \( b \) fluctuation \( \sim \) holon density fluctuation
  \( \sim \) charge density fluctuation.

- Long-wavelength \( b \) fluctuation cost \( \mathcal{E} \downarrow 0 \) if real EM-field is “turned off.”
  \( \implies \) a single linearly-dispersing mode
  \( \sim \) Goldstone boson.

- This time the Goldstone boson is eaten up by the EM-field to produce a superconductor.
  - This superconducting state breaks \( T \)-invariance.

- Intuitively, four species of holon bound together \( \implies \) expects \( \Phi_{EM} = \frac{hc}{4e} \) for a minimal vortex.
  - Confirmed by Chern–Simons formulation.
Hole Doping

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Quasiparticle Statistics—Intuitions

• Quasiparticles are bound states of semions in holon sectors and/or fermions in spinon sector.
  • For finite energy, bound states must be neutral w.r.t. the gapless mode.

• There are two types of “elementary” quasiparticles:
  1. semion-antisemion pair in holon sector;
  2. spinon-holon pair
     (∼ Bogoliubov q.p. in conventional SC).

• All other quasiparticles can be built from these elementary ones.

• Statistics can be derived by treating different species as uncorrelated.
  • Semions from holon sector \(\implies\) existence of semionic (mutual) statistics.
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Crystal momenta of quasiparticle—projective symmetry group study

- For spinon-holon pair, $\mathbf{k}$ is well-defined on the original Brillouin zone.
  - These can be recovered using Projective symmetry group (PSG).

- In contrast, the semion is fractionalized from a holon.
  $\implies$ semion-antimion pair may not have well-defined $\mathbf{k}$. 
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  \[ \implies \text{semion-antisemion pair may not have well-defined } k. \]
Conclusions

- The $U(1)$ Dirac spin-liquid state possess many unusual properties and may be experimentally realized in herbertsmithite.
- The Raman signal of the DSL state has a broad background (contributed by spinon-antispinon continuum) and a $1/\omega$ singularity (contributed by collective [gauge] excitations).
- Under external magnetic field, the DSL state forms Landau levels, which corresponds to a XY symmetry broken state with Goldstone boson corresponding to $S_z$ density fluctuation.
- When the DSL state is doped, an analogous mechanism give rise to an Anderson–Higgs scenario and hence superconductivity.
  - But minimal vortices carry $hc/4e$ flux and the system contains exotic quasiparticle having semionic mutual statistics.

Reference: Ran, Ko, Lee, & Wen, PRL 102, 047205 (2009)
Ko, Lee, & Wen, PRB 79, 214502 (2009)
Ko, Liu, Ng, & Lee, PRB 81, 024414 (2010)
Appendix
(a.k.a. hip pocket slides)
## Comparison of ground-state energy estimate

<table>
<thead>
<tr>
<th>Method</th>
<th>Max. Size</th>
<th>Energy</th>
<th>State</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Diag.</td>
<td>36</td>
<td>$-0.43$</td>
<td>—</td>
<td>[1]</td>
</tr>
<tr>
<td>DMRG</td>
<td>192</td>
<td>$-0.4366(7)$</td>
<td>SL</td>
<td>[2]</td>
</tr>
<tr>
<td>VMC</td>
<td>432</td>
<td>$-0.42863(2)$</td>
<td>U(1) Dirac SL</td>
<td>[3]</td>
</tr>
<tr>
<td>Series Expan.</td>
<td>—</td>
<td>$-0.433(1)$</td>
<td>36-site VBS</td>
<td>[4]</td>
</tr>
<tr>
<td>Entang. Renorm.</td>
<td>—</td>
<td>$-0.4316$</td>
<td>36-site VBS</td>
<td>[5]</td>
</tr>
</tbody>
</table>

Magnified band structure of DSL state, with scales

Energy ($\chi J$) vs $k_y$

-2$\pi$/3  2$\pi$/3
-\$\pi$/3  \$\pi$/3

Energy ($\chi J$) vs $k_x$

-3.4  -3.2  -2.8

Brillouin zone

Wing-Ho Ko (MIT)
Raman signals contributed by spinon-antispinon: full scale

\[ E_g \]

\[ A_{1g} \]

\[ A_{2g} \]

\( (\chi J \approx 56 \text{ cm}^{-1}) \)
Chern–Simons description: B-field case

- For the B-field case, introduce two species of gauge fields to describe the current of up/down spins:

\[ J_{\pm}^\mu = \frac{1}{2\pi} \epsilon^{\mu \nu \lambda} \partial_\nu a_{\pm, \lambda} \]

- The Lagrangian in terms of \( \alpha \) and \( a_{\pm} \):

\[ \mathcal{L} = \pm \frac{1}{4\pi} \epsilon^{\mu \nu \lambda} a_{\pm, \mu} \partial_\nu a_{\pm, \lambda} + \frac{1}{2\pi} \epsilon^{\mu \nu \lambda} \alpha_{\mu} \partial_\nu a_{\pm, \lambda} + \ldots \]

  - higher derivative terms (e.g., Maxwell term \( \sim \partial a \partial a \) for \( a_{\pm} \)) omitted
  - \( \mathcal{L} \) yields the correct equation of motion \( J_{\pm}^\mu = \pm \frac{1}{2\pi} \epsilon^{\mu \nu \lambda} \partial_\nu \alpha_{\lambda} \)

- Let \( c = (\alpha, a_+, a_-)^T \), can rewrite \( \mathcal{L} \) as

\[ \mathcal{L} = -\frac{1}{4\pi} \epsilon^{\mu \nu \lambda} c_\mu^T K \partial_\nu c_\lambda + \ldots \]

  - \( K \) has one null vector \( c_0 \sim \) gapless mode argued previously.
  - Dynamics of \( c_0 \) is driven by Maxwell term \( \Rightarrow \) linearly dispersing.
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Chern–Simons description: doped case

- In the doped case:

\[
\mathcal{L} = -\frac{1}{4\pi} \epsilon^\mu\nu\lambda c_\mu^T K \partial_\nu c_\lambda + \frac{e}{2\pi} \epsilon^\mu\nu\lambda (q \cdot c_\mu) \partial_\nu A_\lambda + (\ell \cdot c_\mu) j^\mu_V + \ldots
\]

where \( c = [\alpha; a_1, \ldots a_4, a_5, a_6; b_1, \ldots b_4] \)

- \( K \) describes the self-dynamics of the system; has null vector \( p_0 = [2; -2, \ldots, -2, 2, 2; 1, \ldots, 1] \) corresponding to gapless mode \( c_0 \).
- \( q = [0; 0 \ldots 0, 0, 0; 1, \ldots, 1] \) is the “charge vector.”
- \( \ell \) is an integer vector with 0 \( \alpha \)-component and characterizes vortices.

- Varying \( \mathcal{L} \) w.r.t. \( c_0 \) gives \( B = -\frac{2\pi}{e} \frac{\ell \cdot p_0}{q \cdot p_0} j^0_V = -\frac{2n\pi}{4e} j^0_V \)
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- Varying \( \mathcal{L} \) w.r.t. \( c_0 \) gives \( B = -\frac{2\pi}{e} \frac{\ell \cdot p_0}{\mathbf{q} \cdot p_0} j_\nu^0 = -\frac{2n\pi}{4e} j_\nu^0 \)
Quasiparticles and their statistics

\[ \mathcal{L} = -\frac{1}{4\pi} \epsilon^{\mu\nu\lambda} c^T \mu K \partial_\nu c_\lambda + \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} (\mathbf{q} \cdot c_\mu) \partial_\nu A_\lambda + (\mathbf{l} \cdot c_\mu) j^\mu_V + \ldots \]

- Vortices with \( \mathbf{l} \cdot \mathbf{p}_0 = 0 \) does not couple to \( c_0 \)
  \( \implies \) They can exist when \( B = 0 \) and corresponds to quasiparticles.

- When particle \( \mathbf{l}_1 \) winds around another particle \( \mathbf{l}_2 \), the statistical phase \( \theta = 2\pi \mathbf{l}_1^T K_\perp^{-1} \mathbf{l}_2 \)
  - \( K_\perp \) is part of \( K \) that’s \( \perp \mathbf{p}_0 \).
  - Derived by integrating out all gauge fields having non-zero Chern–Simons term.

- Taking \( \left\{ \begin{array}{l} \mathbf{l}_1 = [0; 0, \ldots, 0, 0, 0; 1, 0, 0, -1] \\ \mathbf{l}_2 = [0; 0, \ldots, 0, 0, 0; 1, 0, -1, 0] \end{array} \right\} \), found:
  \[ \theta_{11} = \theta_{22} = 2\pi, \quad \theta_{12} = \pi \]
  \( \implies \) Fermions with semionic statistics!
### The full form of $K$-matrix for doped case

The $K$-matrix for the doped case is given by:

$$
K = \begin{pmatrix}
0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{pmatrix}
$$

(Recall $c = [\alpha; a_1, \ldots, a_4, a_5, a_6; b_1, \ldots, b_4]$)

The matrix $K$ is used to describe the interactions in the system. The elements of $K$ correspond to the interactions between different states in the system. In this case, the matrix represents the interactions in the spinon, spinon*, and holon sectors.
Spinon and holon PSG

\[ T_x[\varphi_1^2] = \varphi_4^2 \]
\[ T_x[\varphi_2^2] = \varphi_3^2 \]
\[ T_x[\varphi_3^2] = e^{-\frac{i\pi}{3}} \varphi_2^2 \]
\[ T_x[\varphi_4^2] = e^{\frac{i\pi}{3}} \varphi_1^2 \]

\[ T_x[\eta_1] = e^{\frac{i\pi}{12}} \eta_2 \]
\[ T_x[\eta_2] = e^{\frac{11i\pi}{12}} \eta_1 \]
\[ T_x[\eta_3] = e^{\frac{i\pi}{12}} \eta_4 \]
\[ T_x[\eta_4] = e^{\frac{11i\pi}{12}} \eta_3 \]