Duality walls in 5d gauge theories

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Based on
arXiv:1506.03871 with Davide Gaiotto (Perimeter Institute)
A large class of BPS domain walls has been studied in 4d maximal SUSY gauge theories.

- AdS/CFT, Boundary conditions, S-duality, Branes, …

[Bak, Gutperle, Hirano 03], [Clark, Freedman, Karch, Schnabl 04], [Clark, Karch 05], [D'Hoker, Estes, Gutperle 07], [Bak, Gutperle, Hirano 07], [Gaiotto, Witten 08], …

We are interested in the BPS domain walls in 5d N=1 gauge theories.

We focus on Janus-like domain walls (or interfaces)

- Coupling or mass parameter varies as a function of coordinate.
We consider certain 5d SUSY theories which have CFT fixed points in UV and have relevant deformations to SYMs in IR.

- **Duality domain wall**:

\[ (m, g^2) \leftrightarrow (\tilde{m}, \tilde{g}^2) \]
Introduction

We will propose duality walls, which involve
- boundary conditions
- new 4d degrees of freedom
- 4d superpotentials
- test through explicit partition functions.

We expect to learn
- non-perturivative dynamics at UV fixed point from IR physics.
- close relation between 5d duality and 4d duality.
- new dualities.
1. Introduction.

2. Duality walls in SU(N) gauge theories.

3. Test with partition functions.

4. Duality walls in SU(N) with flavours.

5. Sp(N) $\leftrightarrow$ SU(N+1) duality and domain walls.

6. Conclusion
$\mathcal{N} = 1$ gauge theories in 5d

- Vector multiplet $(A_\mu, \phi; \lambda)$
- Hypermultiplet $(q^A; \psi)$
- Preserve 8 SUSY

There is a topological $U(1)_I$ associated to instanton number symmetry:

$$J_I = \star \text{Tr} F \wedge F$$

Thus, full symmetry is

- $SO(5)$ Lorentz symmetry times $SU(2)_R$ R-symmetry.
- $G$: gauge symmetry.
- $G_F \times U(1)_I$: flavour symmetry.
Basics of 5d SUSY gauge theories

5d gauge theories are non-renormalizable. However, for certain SUSY theories, we expect non-trivial UV fixed points exist.

- QFT analysis
- Branes and string duality, (p,q) five-brane web
- M-theory on CY3

Effective gauge coupling is 1-loop exact: 
\[ \frac{1}{g_{eff}^2} = \frac{1}{g_0^2} + c|\phi| \]

Note that, when \( c > 0 \), we can remove a scale by \( g_0 \rightarrow \infty \) and interacting CFT fixed point can be attained at \( \phi \rightarrow 0 \).

Some UV fixed points enjoy global symmetry enhancement.

(Ex: \( SU(2) \), \( N_f = 5, 6, 7 \) have enhanced \( E_6, E_7, E_8 \) symmetries)
Duality walls in SU(N) theories
\[ \mathcal{N} = 1 \ SU(N)_N \] gauge theory

Five-brane web construction
(at the origin of Coulomb branch)

• \( SU(N) \) gauge theory on \( N \) D5-branes with classical Chern-Simons coupling \( \kappa = N \).

\[
L = \frac{1}{g^2} F \wedge *F + \frac{\kappa}{24\pi^2} A \wedge F \wedge F + \cdots
\]

• Instanton symmetry \( U(1)_I \) is enhanced to \( SU(2) \) at UV fixed point, which comes from two parallel NS5-branes.
5d Duality in IR gauge theories

Mass deformation of UV CFT leads to different IR gauge theories.

We propose a 1/2-BPS domain wall connecting IR dual gauge theories.

Weyl reflection in SU(2) 
$(m \leftrightarrow -m)$

$m \sim g_1^{-2} > 0$

$-m \sim g_2^{-2} > 0$

IR duality 
$(g_1^{-2} \leftrightarrow g_2^{-2})$

We propose a 1/2-BPS domain wall connecting IR dual gauge theories.
Boundary condition and boundary d.o.f

Neumann boundary condition at the interface:
- $F_{5i}|_{\partial} = 0$
- Half-BPS
- Gauge symmetry survives at the boundary

We then couple new 4d degrees of freedom:
- 4d $\mathcal{N} = 1$ matter content:

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<thead>
<tr>
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<th>$SU(N)_l$</th>
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- Superpotential: $W = b \det q$

Consistency requires that boundary gauge anomaly must be cancelled.
Boundary condition and boundary d.o.f

- **4d \( \mathcal{N} = 1 \) matter content**:

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- **Strong constraints by anomaly cancellation**

  1. Cubic anomaly (of unit \( N \)) from 4d matters is cancelled by bulk classical Chern-Simons term at \( \kappa = N \).
  2. Boundary \( U(1)_R \subset SU(2)_R \) is fixed by mixed ’t Hooft anomaly.
  3. Anomaly-free \( U(1)_\lambda \subset U(1)_B \times U(1)_{I_l} \times U(1)_{I_r} \) glues instanton symmetries in both sides.

<table>
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<tr>
<th></th>
<th>( U(1)_\lambda )</th>
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<th>( U(1)_{I_l} )</th>
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Anomaly-free $U(1)_\lambda$ glues together two instanton symmetries on two sides of the wall with opposite signs.

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Therefore, duality wall exchanges gauge couplings

\[
\frac{1}{g^2} \sim m
\]

\[
\frac{1}{\tilde{g}^2} \sim -m
\]

* Duality wall implements $\mathbb{Z}_2$ action in SU(2) global symmetry of UV CFT.
Consistency check:

- 4d theory is now $SU(N)$ SQCD with $N_f = N$ and
  
  $W = b \det q + \tilde{b} \det \tilde{q}$

- "Seiberg dual" theory consists of a meson $M = \tilde{q}q$ and baryons
  
  $B = \det q, \tilde{B} = \det \tilde{q}$ with a constraint
  
  $\det M - B\tilde{B} = \Lambda^{2N}$ and
  
  superpotential: $W = bB + \tilde{b}\tilde{B}$.

\(\text{Trivial interface}\)
SUSY indices with Duality walls
We now see a more non-trivial check with supersymmetric indices in the presence of the duality wall.

- **Superconformal index (SCI)**

\[
I(w_a, q; p, q) = \text{Tr}(-1)^F p^{j_1 + R} q^{j_2 + R} \prod_a w_a^{F_a} q^k
\]

- \( j_1, j_2, R \) are Cartan generators of \( SO(2, 5) \times SU(2)_R \).
- \( F_a \) are Cartans of flavour symm. and \( k \) is instanton number.
- SCI is equivalent to twisted partition function on \( S^1 \times S^4 \).

[S.-S Kim, H.-C Kim, K. Lee 12], [Terashima 12]

- SCI factorizes into two “hemisphere” indices by localization.

\[
I(w_a, q; p, q) = \langle II|II \rangle = \int d\mu_z II(z, w_a, q; p, q) II(z, w_a, q; p, q)
\]

II = \( Z_{\text{pert}} \cdot Z_{\text{inst}} \) : Hemisphere index

\( Z_{\text{pert}} \) = Partition function on \( S^1 \times \mathbb{R}^4 \) with Omega deformation

\( z \) : gauge holonomy
We can insert a duality wall at the equator (with 1/2-SUSY)

\[ I = \langle II(q^{-1})|I^{4d}(q)|II(q)\rangle = \int d\mu_z d\mu_{z'} II(z, q^{-1}; p, q) I^{4d}(z, z', q; p, q) II(z', q; p, q) \]

where \( I^{4d}(z, z', q; p, q) \) is the contribution from 4d d.o.f at the interface (which also depends on the boundary condition).
Duality wall action on hemisphere index

Contribution from 4d d.o.f at interface

<table>
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<tr>
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$\implies I^{4d} = \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i / z'_j)}{\Gamma(\lambda)}$

$z_i : SU(N)_t$

$z'_j : SU(N)_r$

$\lambda : U(1)_\lambda$

( $\Gamma(x)$ : Elliptic gamma function )

We can couple this 4d index to hemisphere index :

$$\hat{DII}^N(z, \lambda) \equiv \int \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \prod_{i,j}^{N} \frac{I^{4d}(z, z', \lambda)}{\Gamma(z'_i / z'_j)} II^N(z'_i, \lambda)$$

• Here, we identify $U(1)_\lambda$ fugacity with instanton number fugacity (or gauge coupling) as $q = \lambda$. 
Duality wall action on hemisphere index

Contribution from 4d d.o.f at interface

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$\Rightarrow \quad I^{4d} = \prod_{i,j=1}^{N} \frac{\Gamma(\lambda^{1/N} z_i / z'_j)}{\Gamma(\lambda)}$

$z_i : SU(N)_l \quad z'_j : SU(N)_r \quad \lambda : U(1)_\lambda$

( $\Gamma(x)$ : Elliptic gamma function )

We can couple this 4d index to hemisphere index :

$$\hat{D}II^N(z, \lambda) \equiv \int \prod_{i=1}^{N-1} \frac{d z'_i}{2\pi i z'_i} \frac{I^{4d}(z, z', \lambda)}{\prod_{i,j} \Gamma(z'_i / z'_j)} II^N(z'_i, \lambda)$$

Duality wall is conjectured to exchange the gauge coupling, therefore, we claim that

$$\hat{D}II^N(z_i, \lambda) = II^N(z_i, \lambda^{-1})$$
Duality wall action on hemisphere index

- Duality wall: \( \hat{D} III^N(z_i, \lambda) = II^N(z_i, \lambda^{-1}) \) [Gaiotto, H.-C Kim 15]

- The hemisphere index is actually given by a series expansion in instanton number. Thus, this is a very surprising claim since the index \( II^N(z, \lambda) \) is expanded by \( \lambda^{k \geq 0} \), while the dual index \( \hat{D} II^N(z, \lambda) \) is expanded by \( (\lambda^{-1})^{k \geq 0} \).

- Can be checked in \( x \equiv (pq)^{1/2} \) expansion.
  - Numerical checks for \( N = 2, 3, 4 \) at least up to \( x^4 \) order.

- More surprisingly, assuming \( II = Z_{\text{pert}} \cdot Z_{\text{inst}} = Z_{\text{pert}} \cdot (1 + O(x)) \), the integral equation

\[
\int \prod_{i=1}^{N-1} \frac{dz_i'}{2\pi i z_i'} \frac{I_{4d}(z, z', \lambda)}{\prod_{i,j}^{N} \Gamma(z_i'/z_j')} II^N(z_i', \lambda) = II^N(z, \lambda^{-1})
\]

uniquely determines the instanton partition function \( Z_{\text{inst}} \) in \( x \) expansion!!
Duality wall action on hemisphere index

• Analytic proof of \( \hat{D}^2 = I \)

\[
\begin{array}{ccc}
\text{SU(N)} & \bullet & \text{SU(N)} \\
\text{SU(N)} & \bullet & \text{SU(N)} \\
\end{array}
\]

= \begin{array}{cc}
\text{SU(N)} & \text{SU(N)} \\
\end{array}

• There is an integral formula (elliptic Fourier transform) \cite{Spiridonov,Warnaar 04}

\[
\int d\mu_{z'} \frac{\prod_{i,j}^N \Gamma(\lambda^{1/N} z_i' / z_j')}{\Gamma(\lambda) \prod_{i,j}^N \Gamma(z_i'/z_j')} \int d\mu_{z''} \frac{\prod_{i,j}^N \Gamma(\lambda^{-1/N} z_i'/z_j'')}{\Gamma(\lambda^{-1}) \prod_{i,j}^N \Gamma(z_i''/z_j'')} f(z'') \sim f(z)
\]

(note: \( R^{4d} = \frac{\prod_{i,j=1}^N \Gamma(\lambda^{1/N} z_i/z_j')}{\Gamma(\lambda)} \))

• This proves \( \hat{D}\hat{D} II(z, \lambda) = II(z, \lambda) \).
Duality walls with flavours
SU(N) gauge theory with flavours

Five-brane web construction

- $SU(N)$ gauge theory with CS coupling at $\kappa = N - N_f/2$.

- IR gauge coupling is identified as $g^{-2} \sim m + \frac{N_f}{2} m_B$, where $m_B$ is the mass parameter for the overall $U(1)_f \subset U(N_f)$ flavor symmetry.

- UV fixed point has an enhanced $SU(2)$ global symmetry and $m$ is the corresponding mass deformation.

We propose a duality interface which exchanges $m \leftrightarrow -m$. 
Boundary conditions and domain wall

Boundary conditions:
- Vector multiplet: Neumann b.c. \( F_{5i}|_\partial = 0 \)
- Hypermultiplet \( \Phi = (X, Y) : X|_\partial = 0, \partial_5 Y|_\partial = 0 \)

We couple this to the same 4d \( \mathcal{N} = 1 \) system
- matter content:

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- Superpotential: \( W = b \det q + Y q X' \)
Duality wall with flavours

- 4d $\mathcal{N} = 1$ matters:

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<td>0</td>
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</tr>
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- Superpotential: $W = b \det q + Y q X'$

Cubic anomaly $N - N_f/2$ at the interface is cancelled by bulk CS-term.

We find anomaly free $U(1)$ global symmetries as (in terms of fugacities)

$$
\begin{array}{cccccc}
\text{fugacity} & q & I_l & I_r & X & X' \\
\lambda^{1/N} & \lambda w^{-N_f/2} & \lambda^{-1} (w')^{-N_f/2} & w & w' \\
\end{array}
$$

( with $U(1)$ fugacities $w = \lambda^{1/N} w'$ and $e^{-\frac{4\pi^2}{g^2}} = \lambda w^{-N_f/2}$, $e^{-\frac{4\pi^2}{(g')^2}} = \lambda^{-1} (w')^{-N_f/2}$, $e^{m_B} = w$, $e^{m'_B} = w'$)
Duality wall action on hemisphere index

Hemisphere index of the boundary condition \( F_{ij} |_{\partial} = 0 \), \( X |_{\partial} = 0 \), \( Y |_{\partial} \neq 0 \)

\[
II^{N,N_f}(z_i, w_a, q; p, q) = \frac{(pq; p, q)^{N-1}_\infty \prod_{i \neq j}^{N} (pqz_i/z_j; p, q)_\infty}{\prod_{i=1}^{N} \prod_{a=1}^{N_f} (\sqrt{pq}z_i/w_a; p, q)_\infty} Z_{\text{inst}}^{N,N_f}(z_i, w_a, q; p, q)
\]

from hypermultiplet \((x; p, q)_\infty : q\text{-Pochhammer symbol}\)

Duality wall action on the hemisphere index

\[
\hat{D} II^{N,N_f}(z, w, \lambda) \equiv \oint \prod_{i=1}^{N-1} \frac{dz_i'}{2\pi i z_i'} \frac{I^{4d}(z, z', \lambda)}{\prod_{i,j}^{N} \Gamma(z_i'/z_j')} II^{N,N_f}(z_i', w, \lambda)
\]

We claim that \( \hat{D} II^{N,N_f}(z_i, w, \lambda) = II^{N,N_f}(z_i, w', \lambda^{-1}) \) (with \( w = \lambda^{1/N} w' \))

- \( \hat{D} : \lambda \rightarrow \lambda^{-1} \)
- Numerical checks for several small \( N, N_f \)

Again, this integral relation of duality wall uniquely determines the full instanton partition function with fund. hypers in \( x = (pq)^{1/2} \) expansion.
Symmetry enhancement and 4d duality

- Example: $SU(2)$ gauge theory with $N_f = 2$ flavours which has symmetry enhancement $SO(4) \times U(1)_I \to SU(2) \times SU(3)$ at the UV fixed point. [Seiberg 96]

- Enhanced $SU(3)$ involves $S_3$ permutation group which exchanges $U(1)_B \times U(1)_I \subset SU(3)$ charges.

- Combinations of duality walls can realize full $S_3$ permutation group.

Let's define two different duality walls with two different b.c. for the hypermultiplets $\Phi_{a=1,2} = (Q_a, \tilde{Q}_a)$

\[ \hat{D}_1: \]

\[ \hat{D}_2: \]

\[ (Q_a: \uparrow, \tilde{Q}_a: \downarrow) \]
Symmetry enhancement and 4d duality

• Concatenation of two domain walls and 4d Seiberg duality shows

\[ \hat{D}_1 \hat{D}_2 = \hat{D}_2 \hat{D}_3 \text{ (or } \hat{D}_3 \hat{D}_1) \]

( where \( \hat{D}_3 : (Q_a, \tilde{Q}_a) \leftrightarrow (\tilde{Q}_a, Q_a) \) )

• Therefore, duality wall actions (with help of 4d Seiberg duality) implement Weyl permutations \( \hat{D}_1, \hat{D}_2, \hat{D}_3 \subset S_3 \) in the \( SU(3) \) at the UV fixed point.
Sp(N) and SU(N+1) duality
Duality between Sp(N) and SU(N+1) theories

Duality between
1. \( Sp(N) \) gauge theory with \( N_f \) fundamental hypers.
2. \( SU(N + 1) \) gauge theory with \( N_f \) fundamental hypers at CS-level

\[ \kappa = N + 3 - \frac{N_f}{2} \quad (N_f < 2N + 6) \]

- \[ \text{Hayashi, S.-S Kim, K. Lee, Taki, Yagi 15, Gaiotto, H.-C Kim 15} \]

Same dimension of Coulomb branch: \( \dim \mathcal{M}_{\text{Coulomb}} = N \)

Same global symmetry at UV fixed point: \( SO(2N_f) \times U(1)_I \)

- \( SU(N) \) gauge theory has enhanced global symmetry as

<table>
<thead>
<tr>
<th>( N_f )</th>
<th>( SU(N)_{\pm(N+1-N_f/2)} )</th>
<th>( N_f )</th>
<th>( SU(N)_{\pm(N+2-N_f/2)} )</th>
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</thead>
<tbody>
<tr>
<td>( \leq 2N )</td>
<td>( SU(N_f + 1) \times U(1) )</td>
<td>( \leq 2N + 1 )</td>
<td>( SO(2N_f) \times U(1) )</td>
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<tr>
<td>( 2N + 1 )</td>
<td>( SU(N_f + 1) \times SU(2) )</td>
<td>( 2N + 2 )</td>
<td>( SO(2N_f) \times SU(2) )</td>
</tr>
<tr>
<td>( 2N + 2 )</td>
<td>( SU(N_f + 2) )</td>
<td>( 2N + 3 )</td>
<td>( SO(2N_f + 2) )</td>
</tr>
</tbody>
</table>

- Can be seen from 1-instanton analysis \[ \text{Yonekura 15, Gaiotto, H.-C Kim 15} \]
- Or from \( (p,q) \) 5-Branes \[ \text{Bergman, Zafrir 14, 15, Hayashi, S.-S Kim, K. Lee, Taki, Yagi 15} \]
We propose a duality wall:

\[
\begin{align*}
\text{Sp}(N), N_f & \quad \text{SU}(N+1)_{N+3-N_f/2}, N_f \\
\end{align*}
\]

4d domain wall

We use a similar boundary conditions

\[
F_{5i}|_{\partial} = 0, \quad X|_{\partial} = 0, \quad Y|_{\partial} \neq 0
\]

And couple it to 4d degrees of freedom at the interface

- 4d \( \mathcal{N} = 1 \) matter content

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<tr>
<td>( M )</td>
<td>1</td>
<td>( N(N+1)/2 )</td>
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<td>-1</td>
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- Superpotential

\[
W = \text{Tr} \ qMq^Tw + XqX'
\]

- When \( N=1 \), it reduces to duality interface in previous \( \text{SU}(2) \) theory

We propose that this is the duality wall that interpolates \( \text{Sp}(N) \) and \( \text{SU}(N+1) \) gauge theories.
Duality action on the hemisphere index of $Sp(N)$ gauge theory.

$$\hat{DII}^{N_f}_{Sp(N)}(z_i, w_a, q_{Sp}; p, q) = II^{N_f}_{SU(N+1)}(z_i', w_a', q_{SU}; p, q)$$

(with $U(1)$ fugacities $w_a = \lambda^{1/2}w_a'$, $q_{Sp} = \lambda^{(N+1)/2}\prod_{a=1}^{N_f}(w_a)^{-1/2}$, $q_{SU} = \lambda^{-1}\prod_{a=1}^{N_f}(w_a')^{-1/2}$).

We claim that

$$\hat{DII}^{N_f}_{Sp(N)} = \int d\mu z_i \frac{I^{4d}(z, z', \lambda)}{\prod_{i>j}(z_i \pm z_j)} \prod_{i=1}^{N_f} \Gamma(z_i^{\pm 2}) II^{N_f}_{Sp(N)}(z_i, q_{Sp}, w_a)$$

Contribution from 4d d.o.f : $I^{4d}(z, z', \lambda) = \prod_{i=1}^{N+1} \prod_{j=1}^{N} \Gamma(\sqrt{\lambda z_i'z_j'})$\]

Checked this relation for $N = 2$ at least up to $x^5$ order.

This integral equation can generate instanton partition functions of $SU(N)_{N^2-2N_f/2}$ gauge theories, which we couldn’t compute using standard ADHM analysis.
(A,C)-type Elliptic integral formula

Concatenation of two duality walls must be a trivial interface: \( \mathring{D} \mathring{D} = I \)

There are (A,C) and (C,A)-type inversion formulas

\[
\oint d\mu_{z'} \Delta^{(A)}(z', x, \lambda) \oint d\mu_z \Delta^{(C)}(z, z', \lambda) f(z) = f(x),
\]
\[
\oint d\mu_z \Delta^{(C)}(z, x, \lambda) \oint d\mu_{z'} \Delta^{(A)}(z', z, \lambda) f(z') = f(x).
\]

\( \Delta^{(A)} \) and \( \Delta^{(C)} \) are the index of 4d d.o.f:

\[
\Delta^{(A)}(z', z, \lambda) \sim \frac{I^{4d}(z, 1/z', 1/\lambda)}{\prod_{i \neq j}^{N+1} \Gamma(z'_i/z'_j)},
\]
\[
\Delta^{(C)}(z, z', \lambda) \sim \frac{I^{4d}(z, z', \lambda)}{\prod_{i > j}^{N} \Gamma(z_i^\pm z_j^\pm) \prod_{i=1}^{N} \Gamma(z_i^{\pm2})}.
\]

• This proves \( \mathring{D} \mathring{D} = I \).
• Duality and domain wall action thus provides a physical interpretation of these elliptic integral identities.

[Spiridonov, Warnaar 04]
Conclusion
Conclusion

• We have proposed duality domain wall connecting two dual SU(N) gauge theories and carried out various tests.

• Enhanced global symmetry in the UV CFT can be seen even in IR gauge theory through the duality wall action and 4d duality at the interface.

• New duality between Sp(N) and SU(N+1) gauge theories and the duality wall between them have been proposed.

Future directions :

• Study on boundary conditions in 5d gauge theories.
• Other duality walls or other type of domain walls.
• Defects in the presence of domain walls.