Topological Order
and Conformal Quantum Critical Points

One might expect that the quasiparticles over a Fermi sea have quantum numbers (charge, spin) of an electron.

This is not always true!
Charge fractionalization occurs in one and two dimensions.

Outline:

1. Spin-charge separation
2. 2+1-dimensional lattice models and field theories
3. Non-abelian symmetries and Chern-Simons theory

work with E. Ardonne and E. Fradkin

related work with R. Moessner and S. Sondhi

and with K. Schoutens, B. Nienhuis and J. de Boer

Charge fractionalization, spin-charge separation and related behavior in two spatial dimensions is of interest for frustrated magnets, topological quantum computation, and high-T_c superconductors.

The low-energy behavior is described by a topological field theory.

Fractional charge has been observed in one spatial dimension by studying the shot noise for tunneling between fractional quantum Hall edges. Theoretically, these particles are the edge modes of a Chern-Simons field theory.

A $1+1$-dimensional example of spin-charge separation familiar to particle physicists is non-abelian bosonization.

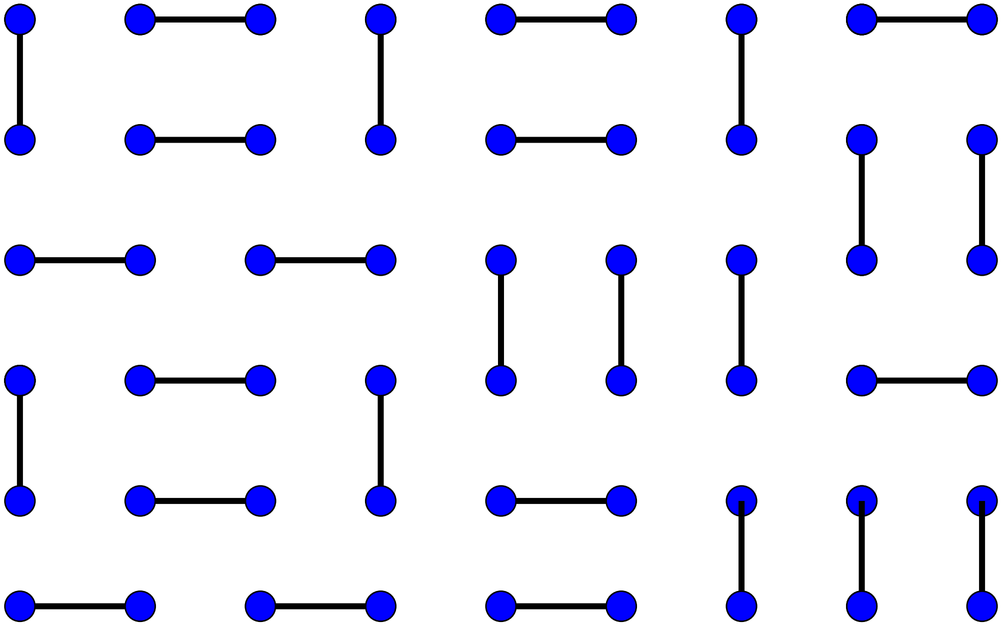
Restrict an electron to move in one dimension. This corresponds to two Dirac fermions in the $1+1$-dimensional field theory, because of the two components of spin. Two free Dirac fermions have symmetry $U(2)$. To bosonize, we have

$$\begin{aligned} \overline{\psi}\sigma^{\mu}\psi &= (\overline{\psi}\vec{\sigma}\psi, \overline{\psi}\psi) \\ U(2) &= SU(2)_1 \times U(1) \\ &= \text{spin} \times \text{charge} \end{aligned}$$

Even though the original theory is made of electrons with both spin and charge, excitations separate into pieces with only spin and only charge.

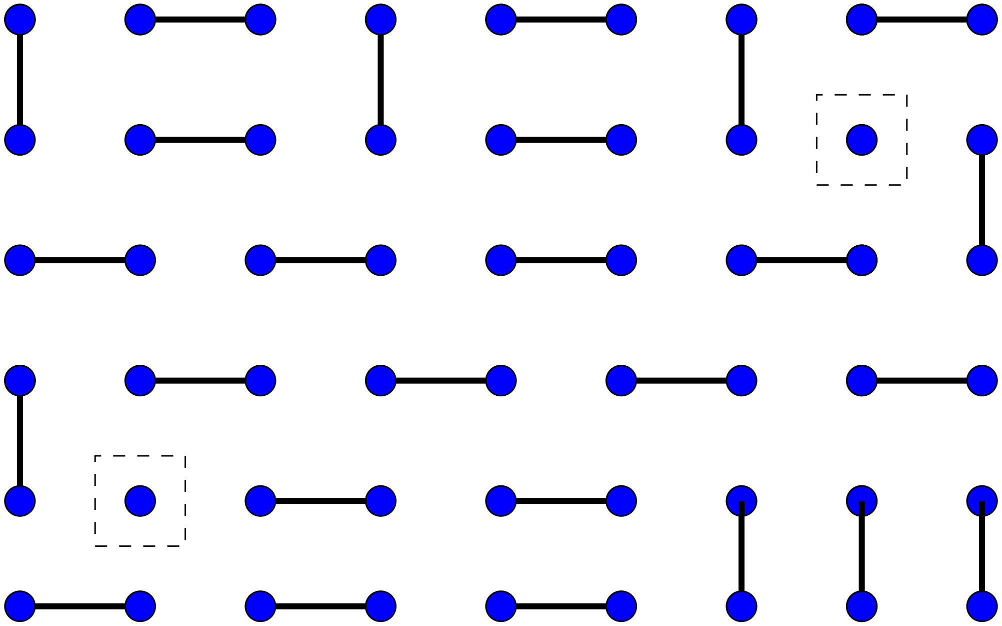
Spin-charge separation in 2d

Electrons (spin 1/2, charge e) on a square lattice, one per site. Say the interactions favor a spin singlet on neighboring sites. The ground state may then be a superposition of "dimer" states like



Each dimer has spin 0 and charge $2e$, so the ground state has spin 0 and charge Ne for N sites. If the state is not ordered, it is called a resonating valence bond state.

To get excited state, break one bond



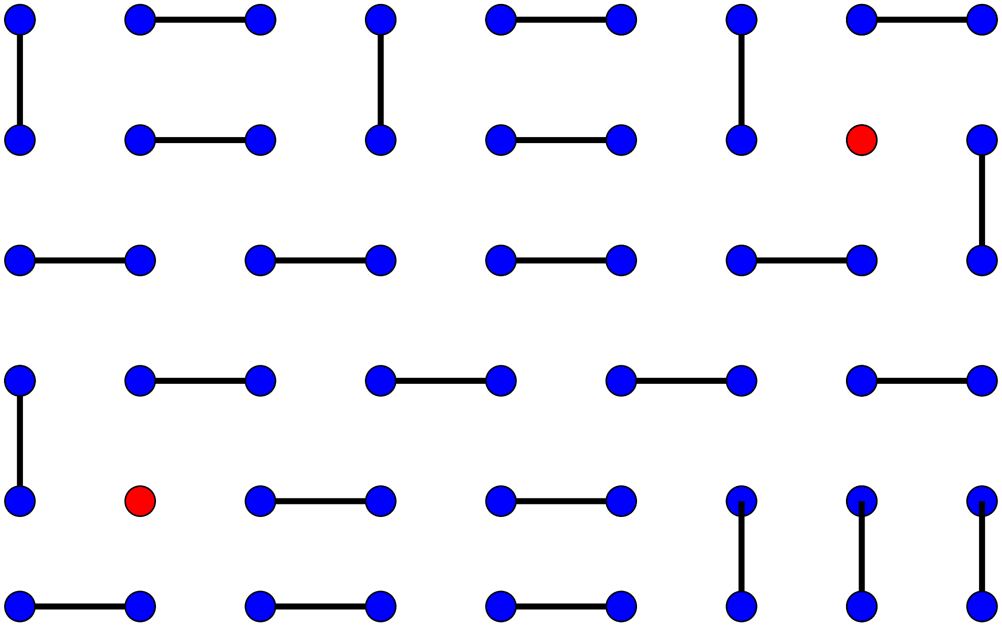
If there is no confinement, you make two quasiparticles. This state has the same charge as the ground state, but different spin.

Thus these particles have charge 0 and spin $1/2$!!

They are called spinons.

Kivelson, Rokhsar and Sethna

Say you remove two electrons, and create a defect:



This two-hole state has no spin, and charge $-2e$ relative to the ground state. Each quasiparticle has charge $-e$ and spin 0 !!

This is like oblique confinement in QCD at $\theta = \pi$: quarks are free, but have no color.

The electron has “split” into a spinon and a holon!

Here's what we would like:

1. A quantum ground state made of a superposition of different states (i.e. a liquid, not an ordered solid)
2. Deconfined and gapped excitations.

There are now concrete models which have this behavior. One finds:

1. The ground state is a topological field theory (correlators do not depend on position).
2. These topological phases are separated from ordered phases by conformal quantum critical points: 2+1 dimensional critical points which exhibit a two-dimensional conformal invariance.
3. In some cases, non-abelian statistics.

The problem: find a quantum Hamiltonian acting on a two-dimensional Hilbert space which has the above properties.

The trick: find a quantum Hamiltonian which has a classical two-dimensional theory describing its ground state. Then we can use the two-dimensional classical results to describe the ground state of the two-dimensional quantum theory.

This has been done for the quantum dimer model.

Rokhsar and Kivelson; Moessner and Sondhi

I will discuss this in field theory, and in a more general lattice model.

Ardonne, Fendley and Fradkin

In field theory:

The continuum limit of the square-lattice quantum dimer model is argued to be 2+1-dimensional scalar field theory with Hamiltonian

$$H = \int d^2x \left[\Pi^2 + \kappa(\nabla^2\phi)^2\right] .$$

where $\Pi = \dot{\phi}$ Henley

This theory arises in three-dimensional classical statistical mechanics as the continuum description of a Lifshitz point, where commensurate, incommensurate and and disordered phases meet.

We thus dub it the quantum Lifshitz theory.

This theory is theory is quadratic in ϕ, and so trivially solvable. However, there is a particularly elegant way of solving it.

The canonical commutation relations are

$$[\phi(\vec{x},t),\Pi(\vec{y},t)] = i\delta^2(\vec{x}-\vec{y}).$$

In the Schrodinger picture, $\Pi = -i\frac{\delta}{\delta\phi}$, giving the Schrodinger equation for the wavefunctional $\Psi[\phi]$:

$$\int d^2x \left[-\left(\frac{\delta}{\delta\phi}\right)^2 + \kappa(\nabla^2\phi)^2 \right] \Psi[\phi] = E\Psi[\phi].$$

Define

$$Q(\vec{x}) = \frac{\delta}{\delta\phi} + \kappa(\nabla^2\phi)$$

and the quantum Hamiltonian is

$$H = \int d^2x \, Q^\dagger Q$$

The ground state has $E = 0$, and the ground-state wave functional satisfies

$$Q(\vec{x})\Psi_{GS}[\phi] = 0.$$

for any $\vec{x}$. This equation is easy to solve, giving

$$\Psi_{GS}[\phi] = e^{-S[\phi]}$$

where

$$S = \frac{\kappa}{2}\int d^2x(\nabla\phi)^2.$$

This is the action of a free two-dimensional boson!

Equal-time correlators in the quantum ground state are those of a classical 2d free-boson field theory

$$\langle e^{i\phi(\vec{x},t)} e^{i\phi(\vec{y},t)} \dots \rangle_{GS} = \frac{\int [D\phi] e^{-2S} e^{i\phi(\vec{x})} e^{i\phi(\vec{y})} \dots}{\int [D\phi] e^{-2S}}$$

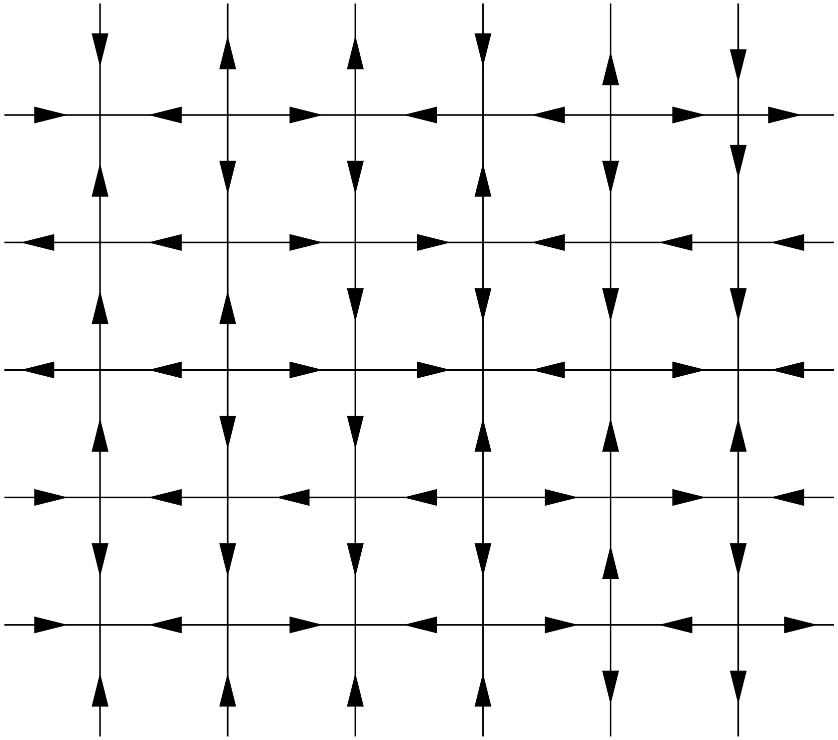
The factor of two is because correlators in a quantum theory are weighted by $|\Psi_0|^2$.

This 2+1-dimensional theory is not Lorentz-invariant. Space and time scale differently: the dynamical critical exponent $z = 2$.

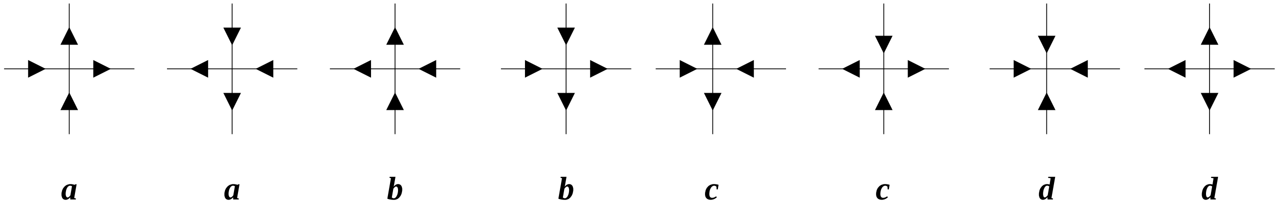
κ parametrizes a conformal quantum critical line. The wave functional is invariant under conformal transformations in space, but not in time.

This critical line is found in the continuum limit of a lattice model, the quantum eight-vertex model:

The basis elements of the Hilbert space of the 2+1-dimensional model are the configurations of the eight-vertex model, e.g.



The classical model assigns Boltzmann weights to each vertex:



We set $a = b = 1$ to preserve rotational symmetry.

We want to find a 2+1-dimensional quantum system from a two-dimensional classical system, like we did for the boson. The ground-state amplitude of a 2+1-dimensional quantum state s is the same as the 2d classical Boltzmann weight:

$$\Psi_0(s) = \frac{e^{-\beta E_s}}{Z(c^2, d^2)}$$

where Z is the classical 2d partition function.

To do this, we use the same trick as for the boson. We construct an operator $\hat{Q}_p$ for each plaquette which annihilates the weighted sum of states. Thus the Hamiltonian

$$H = \sum_p \hat{Q}_p^\dagger \hat{Q}_p$$

has the appropriate properties: any correlator in the quantum ground state is equal to a correlator in the classical 8-vertex model.

We can do this by taking H to be a sum of projection operators, each of which annihilates the weighted sum.

The non-diagonal flip term F_p reverses the arrows around one plaquette. This preserves the restriction that the number of arrows pointing in at any vertex be even.

The potential on each plaquette is then $V_p = c^{n_c - \widetilde{n}_c} d^{n_d - \widetilde{n}_d}$, where n_c and n_d are the number of c and d vertices at the corners of this plaquette, while $\widetilde{n}_c$ and $\widetilde{n}_d$ are the number on the flipped plaquette.

The Hamiltonian $H = \sum_p (V_p - F_p)$ breaks up into 2 by 2 blocks of the form

$$\begin{pmatrix} v & -1 \\ -1 & v^{-1} \end{pmatrix}$$

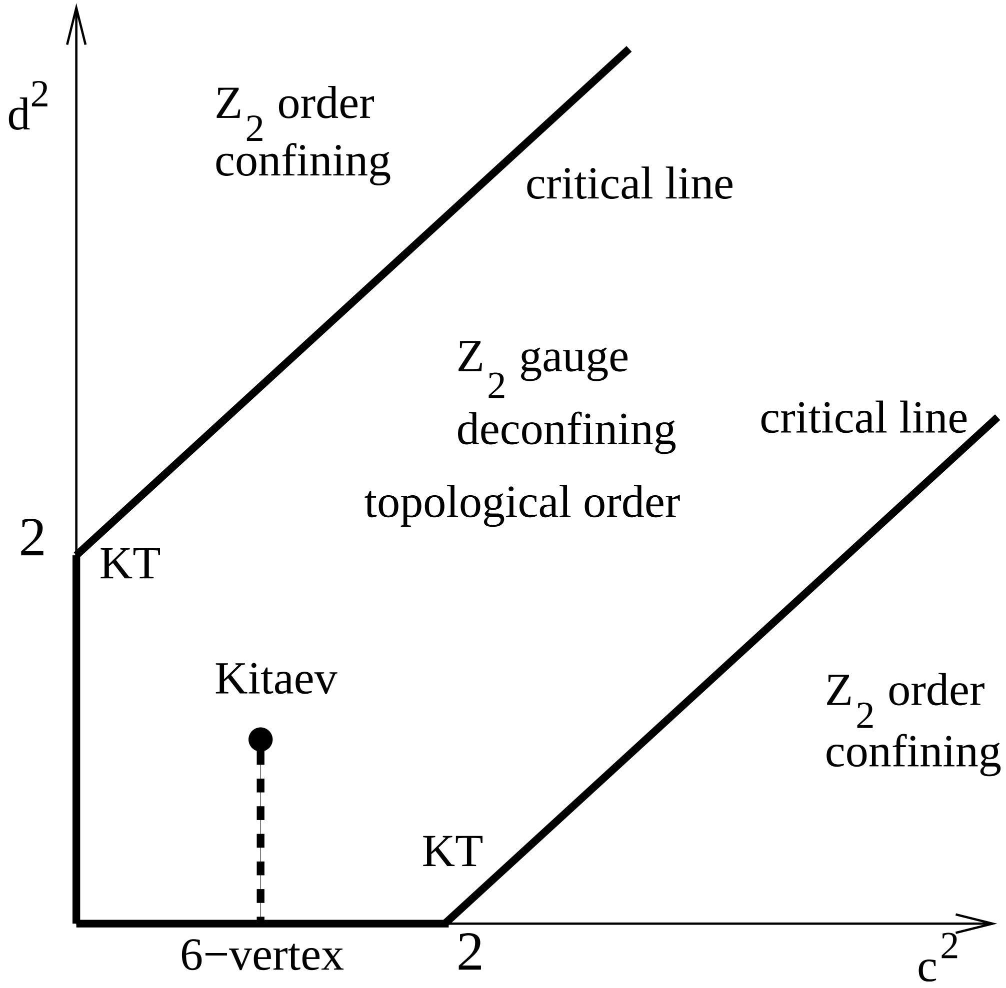
Projectors!

When $a = b = c = d = 1$ (infinite temperature in the classical model), the Hamiltonian is gauge invariant, so this model is a $\mathbb{Z}_2$ lattice gauge theory. Gauge-invariant operators are Wilson loops.

In fact, the ground state is described by a topological field theory.
On genus g, there are 2^g ground states; the flip preserves the number of up arrows mod 2 around a given cycle. Correlation functions of Wilson loops do not depend on distance, but only on how they intersect each other.

This special point is trivial to solve, because every plaquette is independent of the others (no potential). It was originally introduced because the anyonic anyonic excitations (in a generalized version) are valuable for error correction in quantum computers. Kitaev

The 8-vertex model in general is not free-fermionic – it is equivalent to two coupled Ising models. From Baxter's exact results in 2d, we know that the quantum eight-vertex model contains topological order, conventional order, and conformal quantum critical lines.



Field theory can also decribe the topological phase.

The classical eight-vertex model is equivalent to a free-fermion model when $ab = cd$. On the critical line, the fermion is massless; off, it's massive. Moreover,

- The fermion can be bosonized. Off the critical point this is the sine-Gordon model.
- The entire critical line is given by changing the boson radius/coupling κ.
- Sine-Gordon field theory describes the entire region near the critical line.

This means that the continuum limit of the quantum eight-vertex model is related to sine-Gordon as well.

Defining $H = \int Q^\dagger Q$, with

$$Q(x) \equiv \frac{\delta}{\delta\varphi} + \kappa\, \nabla^2\varphi + \frac{\lambda}{2\pi}\sin(2\pi\varphi)$$

The ground-state wave functional is then

$$\Psi_0 = \frac{e^{-S_{SG}}}{Z_{SG}}.$$

Grinstein

What we have done so far:

We have shown that one can construct $2+1$-dimensional models which exhibit both topological phases and conformal quantum critical points with $z=2$.

These models are very special because their ground-state wave functions are directly related to a classical two-dimensional model. This certainly is not generic behavior.

This is generic behavior for topological field theories, however. Topological field theories have a BRST charge Q_{BRST} satisfying $Q_{BRST}^2=0$. The states of the topological field theory are annihilated by Q_{BRST}, and not Q_{BRST} of something else (the cohomology). Q_{BRST} can be written in terms of a local two-dimensional density.

In a topological phase, the ground state is a topological field theory. The rest of the theory is a gapped $2+1$ dimensional theory.

Can we go beyond these abelian theories?

One might think that one could use the same trick on WZW theories:

$$S_{WZW} = \lambda\, \mathrm{tr} \int d^2z\, \partial g \bar{\partial} g^{-1} + ikS_{WZ}$$

where g is in some Lie group G and k is an integer called the “level”.

There is a critical point for $k \neq 0$ and $\lambda = |k|/(16\pi^2)$ known as the G_k WZW model. For $k = 0$ it is called the principal chiral model and is massive.

One can easily find a Hamiltonian with

$$\Psi_0[g] = \frac{1}{Z} e^{-S_{WZW}[g]}$$

But...

The 2d classical theory is critical for $k \neq 0$. The 2+1-dimensional theory is not!

The catch is that states in the 2+1-dimensional quantum theory are weighted by $|\Psi_{GS}|^2$. Because of the i in front of the WZ term, it does not appear in $|\Psi_{GS}|^2$. Thus correlators are evaluated in the (massive) principal chiral model, and so exponentially decay.

A critical theory in 2d does not automatically yield a 2+1d quantum critical point!

We do know that Chern-Simons field theories in 2+1 dimensions are deeply connected to WZW models in 2 dimensions.

Moreover, pure Chern-Simons theory is topological.

We want to find a field theory which has this topological field theory as its ground state. The ground states are Wilson loops (loops in the two-dimensional space). This theory (doubled) is useful for topological quantum computation.

Freedman, Nayak, Shtengel, Walker and Wang

Let's think in terms of two-dimensional wave functionals like before. Couple the WZW model to a gauge field.

$$I(g,A) = S_{WZW}(g) + \frac{k}{4\pi}\int d^2z \,\mathrm{Tr}\left[2A_{\overline{z}}g^{-1}\partial_z g \;-\; A_{\overline{z}}A_z\right]$$

We don't want $e^{-I(g,A)}$ as a wave functional, because the WZ term will cancel in $|\Psi|^2$ like before. Instead, consider

$$\Psi[A] \equiv \frac{1}{Z}\int [Dg] e^{-I(g,A)}.$$

Witten has shown that

$$Z = \int [DA] |\Psi[A]|^2 = \int [Dg] e^{-S_{WZW}(g)}$$

including the i !!

This is not the wavefunction of a critical theory, however. Correlators of gauge-invariant operators are either topological or decay exponentially.

To see this, let us find the $2+1$-dimensional field theory which has $\Psi[A]$ as its ground-state wave functional.

Witten has shown (from the definition of $\Psi[A]$) that

$$\left(\frac{\delta}{\delta A_z} - \frac{k}{4\pi} A_{\overline{z}}\right) \Psi[A] = 0.$$

$$\left(D_{\overline{z}} \frac{\delta}{\delta A_{\overline{z}}} + \frac{k}{4\pi} D_{\overline{z}} A_z - \frac{k}{2\pi} F_{\overline{z}z}\right) \Psi[A] = 0$$

From these equations, one finds that this is the ground-state wave functional of

strongly-coupled Yang-Mills theory with a Chern-Simons term

Grignani, Semenoff, Sodano, Tirkkonen

$$S = S_{CS} + S_{SC}$$

$$S_{CS} = \frac{k}{4\pi} \int_M \epsilon^{\mu\nu\alpha} \text{Tr} \left[A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha \right]$$

$$S_{SC} = \frac{1}{2e^2} \int_M \text{Tr} \left[F_{0i} F^{0i} \right]$$

Pure Chern-Simons theory has Hamiltonian $H = 0$ (it is topological). The Hamiltonian comes purely from the Maxwell term. We have

$$H = e^2 \int d^2x \, \mathrm{Tr} \left[E^\dagger(\vec{x}) E(\vec{x}) \right]$$

where

$$E \equiv \frac{i}{e^2} \left(F_{01} + i F_{02} \right) = \frac{\delta}{\delta A_z} - \frac{k}{4\pi} A_{\overline{z}}$$

Ground states obey $E\Psi_0[A] = 0$. This is Witten's first equation!

The second equation comes from demanding that $\Psi_0[A]$ have the proper gauge-transformation properties – the non-abelian version of Gauss' Law.

Jackiw; Elitzur, Moore, Schwimmer and Seiberg

This is not a critical theory – it has a gap.

The ground states are Wilson loops.

The excited states are Polyakov loops.

The two are not equivalent because the theory is not Lorentz-invariant.

Note that E and $E^\dagger$ are like annihilation/creation operators:

$$[E^a(\vec{x}), (E^b(\vec{y}))^\dagger] = \frac{k}{2\pi}\delta^{ab}\delta(\vec{x}-\vec{y})$$

Polyakov loops are created by acting with an appropriate gauge-invariant integral of E acting on the ground state.

It is technically convenient and physically important to work in the doubled theory. One has two gauge fields A, B and

$$\chi[A,B] \equiv \Psi[A]\overline{\Psi[B]}$$

This obeys

$$\int [DB]\,|\chi[A,B]|^2 = \int [Dg] e^{-I_{G/G}(g,A)}$$

where $I_{G/G}$ is the action of the G/G gauged WZW model, namely

$$I_{G/G}(g,A) = I(g) + \frac{k}{2\pi}\int d^2z\, \text{Tr}\left[A_{\overline{z}} g^{-1}\partial_z g + A_z g \partial_{\overline{z}} g^{-1} - A_{\overline{z}} A_z + A_z g A_{\overline{z}} g^{-1}\right] .$$

As opposed to $I(g,A)$ above, this action is gauge invariant, because one is gauging the full anomaly-free $G_L \times G_R$ symmetry of the WZW model.

This G/G theory is equivalent to the Chern-Simons topological field theory.

Spiegelglas; Witten

We thus have an explicit 2+1-dimensional field theory which has Chern-Simons theory as its ground state.

The full two-dimensional theory is massive; it is equivalent to the (twisted) $\mathcal{N} = 2$ supersymmetric sigma model on the Grassmanian manifold.

Gepner; Vafa; Intrilgator; Witten

We have thus seen that

- There are simple lattice models and field theories which exhibit topological order and conformal quantum critical lines.
- Equal-time correlators of some of these models can be computed exactly.
- There is a gapped field theory with non-abelian statistics.
- There seems to be no non-trival conformal quantum critical point with non-abelian symmetry.

these transparencies at `http://rockpile.phys.virginia.edu`