Symmetry constraints on counterterms in $N = 8$ supergravity

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Based on

arXiv:1009.1643 w/ Niklas Beisert, Dan Freedman, Michael Kiermaier, Alejandro Morales, Stephan Stieberger

arXiv:1007.4813 w/ Michael Kiermaier

Is $\mathcal{N} = 8$ supergravity UV finite in 4d?
Perturbative structure of $\mathcal{N} = 8$ supergravity in 4d

$L$-loop divergence $\leftrightarrow$ counterterm of mass dimension $(2L + 2)$

for example: $R^4$ at 3-loop order
Perturbative structure of $\mathcal{N} = 8$ supergravity in 4d

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Candidate counterterms are

- local operators
- $\mathcal{N} = 8$ SUSY
- $SU(8)_R$-invariant
- $E_7(7)$-compatible
Pure supergravity finite at 1- and 2-loop order.

Purely gravitational operators are contractions of Riemann tensors $R_{\mu\nu\rho\sigma}$ and covariant derivatives $D_\mu$. Here’s the chart:

<table>
<thead>
<tr>
<th>L</th>
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<tr>
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Must require $\mathcal{N} = 8$ SUSY and $SU(8)$. 
Analysis of potential counterterms

Instead of studying the operators, we analyze their matrix elements:

\[
\text{operator} \leftrightarrow \text{matrix elements}
\]

local \leftrightarrow \text{polynomial in momenta and polarizations}
\leftrightarrow \text{polynomial in } \langle ij \rangle \text{ and } [ij].

\[
L\text{-loop} \leftrightarrow \langle ij \rangle, [ij] \text{ polynomial has degree } 2L + 2.
\]

\[
\mathcal{N} = 8 \text{ SUSY} \leftrightarrow \text{SUSY Ward identities.}
\]

\[
SU(8)\text{-invariant} \leftrightarrow SU(8) \text{ Ward identities.}
\]

\[
E_{7(7)}\text{-compatible} \leftrightarrow \text{low-energy theorems}
\]

\textit{no such matrix elements} \leftrightarrow \textit{no such operator} \leftrightarrow \textit{no such counterterm}.

\textit{If matrix elements do exist: determine multiplicities of such operators.}
Outline

1 Part 1: $\mathcal{N} = 8$ SUSY and $SU(8)$.

2 Part 2: $E_{7(7)}$ constraints.

3 The end: “Landscape” of candidate counterterms.
Tool kit

• **“Little group scaling”:**
  For each external state \( i = 1, \ldots, n \),
  
  \[
  |i\rangle \rightarrow t_i|i\rangle \quad \text{and} \quad |i\rangle \rightarrow t_i^{-1}|i\rangle, \quad \Longrightarrow \quad A_n \rightarrow t_i^{-2h_i}A_n
  \]
  where \( h_i \) is the helicity.

• **Dimensional analysis:**
  Each \( \langle ij \rangle \) and \([ij]\) has mass dimension 1.

• \( \mathcal{N} = 4, 8 \) **SUSY Ward identities:**

  MHV: \( \langle + + -- + + + + \ldots \rangle = \frac{\langle 34 \rangle^{\mathcal{N}}}{\langle 12 \rangle^{\mathcal{N}}} \langle -- + + + + + + \ldots \rangle \).

  **Example:** 4-gluon MHV amplitude

  \[
  A_n(1^-2^-3^+4^+\ldots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \cdots \langle n1 \rangle}
  \]
  has mass dim. \( 4 - n \).
Example of how we exclude operators as candidate counterterms.

4-loops: $R^5$ (mass dim. $2L + 2 = 10$)

10 derivatives in $R^5$ $\rightarrow$ leading 5-point interaction has 10 power of momentum
$\rightarrow$ 5-pt matrix element has mass dim. 10
and is polynomial of degree 10 in $\langle .. \rangle$’s and $[..]$’s.
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Little grp scaling → \( \langle 1^-2^-3^+4^+5^+ \rangle_{R^5} \) contains \( \{ |1\rangle^4, |2\rangle^4, |3\rangle^4, |4\rangle^4, |5\rangle^4 \)  

unique: \( \langle 1^-2^-3^+4^+5^+ \rangle_{R^5} = \langle 12 \rangle^4 [34]^2 [45]^2 [53]^2 \)
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SUSY Ward Id.s → $\langle 1^{+2^{+3^{-4^{-5^{+}}}}}_{R^5} \rangle = \frac{\langle 34 \rangle^{8}}{\langle 12 \rangle^{8}} \langle 1^{-2^{-3^{+4^{+5^{+}}}}}_{R^5} \rangle$ i.e.

$\langle 34 \rangle^{4}[12]^{2}[25]^{2}[51]^{2} = \frac{\langle 34 \rangle^{8}}{\langle 12 \rangle^{8}} \langle 12 \rangle^{4}[34]^{2}[45]^{2}[53]^{2}$

local = non-local conflict
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SUSY Ward Id.s $\rightarrow$ $\langle 1^+ 2^+ 3^- 4^- 5^+ \rangle_{R^5} = \frac{\langle 34 \rangle^8}{\langle 12 \rangle^8} \langle 1^- 2^- 3^+ 4^+ 5^+ \rangle_{R^5}$ i.e.

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local = non-local conflict

$\Rightarrow$ No $\mathcal{N} = 8$ SUSY matrix elements. So $R^5$ is not indep. supersymmetrizable.
Analysis

Carry out an analysis of matrix elements at MHV and NMHV level.

[HE, Freedman, Kiermaier, 1003.5018]

- Use superamplitudes.

- Use solution to SUSY Ward identities.
  [HE, Freedman, Kiermaier, 0911.3169]

- Use Gröbner basis.
  [Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]
Chart of potential counterterms

The matrix elements of a prospective counterterm must respect $\mathcal{N} = 8$ SUSY and $SU(8)$ Ward identities.

If no: excluded. If yes: we find multiplicities of such operators.

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"None $\rightarrow$": we proved no MHV and no NMHV, and conjectured no $N^k$ MHV for $L < 7$ in [HE, Freedman, Kiermaier, 1003.5018]. Conjecture proven by [Howe, Heslop, Drummond, 1008.4939].
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Is $R^4$ compatible with $E_{7(7)}$?

To test $E_{7(7)}$ we will need a 6-point matrix element of $R^4$ with two scalars and four gravitons:

$$\langle \varphi \overline{\varphi} ++-- \rangle_{R^4}$$

Very hard to calculate from Feynman diagrams

We use a trick to extract the 6-point $R^4$ matrix elements from the closed string theory tree amplitude.
Outline

1. **Part 1:** $\mathcal{N} = 8$ SUSY and $SU(8)$.

2. **Part 2:** $E_7(7)$ constraints.
   - From open string amplitudes to closed string amplitudes via KLT.
   - String tree amplitudes and their symmetries.
   - $R^4$ and $E_7(7)$.
   - Matching with automorphic function.
   - $E_7(7)$ at higher loop order.

3. “Landscape” of candidate counterterms.
KLT relations in string theory

Kawai-Lewellen-Tye (KLT) relations:

\[
(closed \text{ string tree amplitude}) = \sum f(s) (open \text{ string tree amplitude})_L \times (open \text{ string tree amplitude})_R
\]

for example for 5-point amplitudes

\[
M_5(1, 2, 3, 4, 5) = -\frac{\sin(\alpha'\pi s_{12})\sin(\alpha'\pi s_{34})}{\alpha'^2\pi^2} A_5(1, 2, 3, 4, 5) \tilde{A}_5(2, 1, 4, 3, 5) + (2 \leftrightarrow 3).
\]

The decomposition of states is “closed string = L and R movers”.

In the following:

- Toroidally compactified Type II superstring theory in \( D = 4 \).
- Allow ONLY massless external states.

  - open string states \( \leftrightarrow \) 16 states of \( \mathcal{N} = 4 \) SYM
  - closed string states \( \leftrightarrow \) 256 states of \( \mathcal{N} = 8 \) supergravity
$N = 4$ SYM

$2^4 = 16$ massless states

<table>
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</tr>
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</tr>
<tr>
<td>6 scalars</td>
<td>0</td>
</tr>
<tr>
<td>4 gluinos</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
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$g^{1234} = g^-$

3 pairs of complex scalars are self-conjugate: $\bar{Z}_{ab} = \frac{1}{2} \epsilon_{abcd} z^{cd}$.

Global $SU(4)$ R-symmetry: $A_n(z^{12}, g^-, z^{34}, \ldots) = 0$ unless $SU(4)$-singlet.
\( \mathcal{N} = 8 \) supergravity

\[ 2^8 = 256 \text{ massless states} \]

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<tr>
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</thead>
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<tr>
<td>1 graviton</td>
<td>+2</td>
<td>( h^+ )</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>70 scalars</td>
<td>0</td>
<td>( \varphi^{abcd} )</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>1 graviton</td>
<td>−2</td>
<td>( h^- = h^{12345678} ) ( (a, b, \ldots = 1, \ldots, 8) )</td>
</tr>
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35 pairs of complex scalars are self-conjugate: \( \overline{\varphi}_{abcd} = \frac{1}{4!} \epsilon_{abcdefgh} \varphi_{efgh} \).

\( \mathcal{N} = 8 \) supergravity has global \( SU(8) \) R-symmetry:

\[ M^\text{SUGRA}_n (v^{12}, \varphi^{1245}, \ldots) = 0 \] unless \( SU(8) \)-singlet.
All $2^8 \mathcal{N} = 8$ states decompose into $2^4 \times 2^4 \mathcal{N} = 4$ SYM states.

For example, gravitons = gluon$^2$: $h^\pm = g^\pm \otimes g^\pm$

Where the 35 pairs of complex scalars come from:

Decompose $SU(8) \rightarrow SU(4) \times SU(4)$ as $\{1, \ldots, 8\} \rightarrow \{1, 2, 3, 4\} \otimes \{5, 6, 7, 8\}$

1) 1 pair is $SU(4) \times SU(4)$-singlet
   \[\varphi \equiv \varphi^{1234} = g^{1234} \otimes g^+ = g^- \otimes g^+\]
   \[\bar{\varphi} \equiv \varphi^{5678} = g^+ \otimes g^{5678} = g^+ \otimes g^- .\]

2) 16 pairs $\bar{4} \otimes 4$: $\varphi_\ell = \lambda^- \otimes \lambda^+$ ex. $\varphi^{123|5}$

3) 18 pairs $6 \otimes 6$: $\varphi_s = z \otimes z$ ex. $\varphi^{12|56}$
KLT relations, e.g. with $h^\pm = g^\pm \otimes g^\pm$

$$M_5(1^-, 2^-, 3^+, 4^+, 5^+) = -\frac{\sin(\alpha' \pi s_{12}) \sin(\alpha' \pi s_{34})}{\alpha'^2 \pi^2} A_5(1^-, 2^-, 3^+, 4^+, 5^+) \tilde{A}_5(2^-, 1^-, 4^+, 3^+, 5^+) + (2 \leftrightarrow 3).$$

- KLT makes $SU(4) \times SU(4)$ a manifest global symmetry of the $D=4$ closed string tree amplitudes $M_n$ with massless external states.

- But closed string theory has no global continuous symmetries!

- $SU(4) \times SU(4) \subset T$-duality group $SO(6, 6)$. Global symmetry only in this sector, only at tree level.

- Classification needs two integers $k$ and $\tilde{k}$: $N^{(k, \tilde{k})}$ MHV.
Example of $SU(8)$-violating amplitude

\[ M_5(1^- 2^- 3^+ 4^+ \varphi^{1234}_5) \quad \text{classification } N^{(1,0)}_{\text{MHV}} \equiv "\sqrt{N}_{\text{MHV}}" \]

\[ = - \frac{\sin(\alpha' \pi s_{12}) \sin(\alpha' \pi s_{34})}{\alpha'^2 \pi^2} A_5(1^- 2^- 3^+ 4^+ 5^-) \tilde{A}_5(2^- 1^- 4^+ 3^+ 5^+) + (2 \leftrightarrow 3) \]
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$$= \alpha'^3 \ 6 \zeta(3) \langle 12 \rangle^4 [34]^4 + O(\alpha'^5).$$

This amplitude violates $SU(8)$!! but vanishes for $\alpha' = 0$ as required by $SU(8)$ in supergravity.

\[ \ldots \text{preserves } SU(4) \times SU(4). \]
Example of $SU(8)$-violating amplitude

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\ldots preserves $SU(4) \times SU(4)$.

Let’s try to understand this better:

Note the $\alpha'^3$ matrix element has no poles

\ldots comes from a local operator with 8 derivatives

\ldots candidate: $\varphi R^4$
The first operator in the closed string effective action is (in Einstein frame)
\[ \alpha' \sqrt{-g} e^{-6\phi} R^4 = \alpha' \sqrt{-g} (1 - 6\phi + \ldots) R^4, \]
where \( \phi \) is the dilaton.

Its 4- and 5-point matrix elements are
\[ \langle 1^- 2^- 3^+ 4^+ \rangle_{e^{-6\phi} R^4} = -\alpha'^3 2 \zeta(3) \langle 12 \rangle^4 [34]^4, \]
\[ \langle 1^- 2^- 3^+ 4^+ \phi \rangle_{e^{-6\phi} R^4} = \alpha'^3 12 \zeta(3) \langle 12 \rangle^4 [34]^4. \]
How to identity the dilaton among the 70 scalars of the $\mathcal{N} = 8$ spectrum?

It is $SU(4) \times SU(4)$-invariant and respects L/R exchange:

Recall: 1 pair is $SU(4) \times SU(4)$-singlet

$$\varphi \equiv \varphi^{1234} = g^- \otimes g^+, \quad \overline{\varphi} \equiv \varphi^{5678} = g^+ \otimes g^-.$$

This identifies: $\phi = \frac{1}{2}(\varphi^{1234} + \varphi^{5678}).$

Then

$$\left\langle 1^- 2^- 3^+ 4^+ \phi \right\rangle e^{-6\phi R^4} = M_5 (1^- 2^- 3^+ 4^+ \varphi^{1234})|_{\alpha'^3} + M_5 (1^- 2^- 3^+ 4^+ \varphi^{5678})|_{\alpha'^3}.$$ 

$$12 \zeta(3) \langle 12 \rangle^4 [34]^4 = 6 \zeta(3) \langle 12 \rangle^4 [34]^4 + 6 \zeta(3) \langle 12 \rangle^4 [34]^4$$

So the dilaton is ‘responsible’ for the $SU(8)$-violation.
The $\alpha'^3$-correction to the closed string tree amplitude are encoded in the supersymmetrization of

$$\alpha'^3 \sqrt{-g} e^{-6\phi} R^4$$

This preserves only $SU(4) \times SU(4)$.

The $\alpha'$-corrections explicitly break $SU(8) \rightarrow SU(4) \times SU(4)$ because the dilaton singles out a special “direction” in $SU(8)$.

We cannot use the closed string tree amplitude directly to explore the 3-loop $R^4$ candidate counterterm of $\mathcal{N} = 8$ supergravity, because it has to be an $SU(8)$-invariant supersymmetrization.
\( \mathcal{N} = 8 \) supergravity has a global continuous \( E_{7(7)} \) symmetry which is spontaneously broken to \( SU(8) \).

The \( 133 - 63 = 70 \) scalars are the Goldstone bosons.
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**Low-energy theorems:**

In \( \mathcal{N} = 8 \) supergravity, single soft scalar limits vanish,

\[
M_n(\varphi(p), \ldots) \to 0 \quad \text{as} \quad p \to 0.
\]

[Bianchi, HE, Freedman ’0805; Arkani-Hamed, Cachazo, Kaplan ’0808; Kallosh, Kugo ’0811]
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**Counterterms:**

\( E_{7(7)} \) compatible? Test if the single soft scalar limits of their matrix elements vanish.

Specifically, for \( R^4 \) we would like to calculate

\[
\lim_{p_1 \to 0} \left< \varphi \varphi^\dagger 3^- 4^- 5^+ 6^+ \right>_{R^4} = ?
\]

to test if it vanishes or not.

Earlier work w/ \( e^{-6\phi} R^4 \) [Brödel & Dixon, 2009]

Single soft limits of the MHV 4-, 5- and 6-pt matrix elements trivially vanish
From $e^{-6\phi} R^4$ to $R^4$

How to obtain $R^4$ matrix elements from $\alpha'^3$ of the string amplitude:

‘Average’ the $\alpha'^3$ contributions of the string amplitude over $SU(8)$

$\implies$

‘Average’ the matrix elements of $e^{-6\phi} R^4$ over $SU(8)$

$\implies$

matrix elements of an $SU(8)$-invariant supersymmetric 8-derivative operator.

There is only ONE such operator, namely the desired $R^4$.

[Freedman, Kiermaier, H.E. (March 2010)]
Product of two scalars $\phi^{abcd}$ contains one singlet: $(\varphi \bar{\varphi})_{\text{sing}} = \frac{1}{8!} \epsilon_{abcdefh} \varphi^{abcd} \varphi^{efgh}$.

Thanks to $SU(4) \times SU(4)$, we get

$$\langle \varphi \bar{\varphi} + + -- \rangle_{R^4} = \frac{1}{35} \langle \varphi^{1234} \varphi^{5678} + + -- \rangle_{e^{-6\phi R^4}} - \frac{16}{35} \langle \varphi^{123|5} \varphi^4|678 + + -- \rangle_{e^{-6\phi R^4}}$$

$$+ \frac{18}{35} \langle \varphi^{12|56} \varphi^{34|78} + + -- \rangle_{e^{-6\phi R^4}}.$$
Average of $SU(8)$

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$$
\langle \phi \bar{\phi} + + - - \rangle_{R^4} = \frac{1}{35} \langle \phi^{1234} \phi^{5678} + + - - \rangle_{e^{-6\phi R^4}} - \frac{16}{35} \langle \phi^{1235} \phi^{4678} + + - - \rangle_{e^{-6\phi R^4}} \\
+ \frac{18}{35} \langle \phi^{12} | 56 \phi^{34} | 78 + + - - \rangle_{e^{-6\phi R^4}}.
$$

We calculate these 3 matrix elements from the $\alpha'$-expansion of the closed string NMHV amplitudes, obtained via KLT

($\alpha'$-expansion of open string amplitude from Stieberger & Taylor)
Average of $SU(8)$

Product of two scalars $\phi^{abcd}$ contains one singlet: $(\varphi \overline{\varphi})_{\text{sing}} = \frac{1}{8!} \epsilon_{abcdefg} \phi^{abcd} \varphi^{efgh}$.

Thanks to $SU(4) \times SU(4)$, we get

$$\langle \varphi \overline{\varphi} + + - - \rangle_{R^4} = \frac{1}{35} \langle \varphi^{1234} \varphi^{5678} + + - - \rangle e^{-6\phi R^4} - \frac{16}{35} \langle \varphi^{1235|4} \varphi^{4|678} + + - - \rangle e^{-6\phi R^4}$$

$$+ \frac{18}{35} \langle \varphi^{12|56} \varphi^{34|78} + + - - \rangle e^{-6\phi R^4}.$$  

We calculate these 3 matrix elements from the $\alpha'$-expansion of the closed string NMHV amplitudes, obtained via KLT ( \(\alpha'\)-expansion of open string amplitude from Stieberger & Taylor)

- \(\lim_{p_1 \to 0} \langle \varphi^{1234} \varphi^{5678} + + - - \rangle e^{-6\phi R^4} = -12 \zeta(3) \times [34]^4 \langle 56 \rangle^4\),
- \(\lim_{p_1 \to 0} \langle \varphi^{1235|4} \varphi^{4|678} + + - - \rangle e^{-6\phi R^4} = -6 \zeta(3) \times [34]^4 \langle 56 \rangle^4\),
- \(\lim_{p_1 \to 0} \langle \varphi^{12|56} \varphi^{34|78} + + - - \rangle e^{-6\phi R^4} = 0\).

hence

\[
\lim_{p_1 \to 0} \langle \varphi \overline{\varphi} + + - - \rangle_{R^4} = 2\zeta(3) \frac{6}{5} [34]^4 \langle 56 \rangle^4 \neq 0.
\]

Conclusion: the unique $SU(8)$-invariant supersymmetrization of $R^4$ is NOT $E_{7(7)}$-compatible.
Candidate counterterm operators must be $\mathcal{N} = 8$ SUSY and $SU(8)$-invariant and have $E_{7(7)}$ symmetry.

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Observation 1

\[(\star) \quad \text{Why} \quad \lim_{p_1 \to 0} \left< \varphi^{12|56} \varphi^{34|78} + + - - \right> e^{-6\phi R^4} = 0? \]

- \( \mathcal{N} = 8 \) supergravity:
  
  Global \( E_{7(7)} \) symmetry spontaneously broken to \( SU(8) \).
  
  The \( 133 - 63 = 70 \) scalars are the Goldstone bosons, which transform in the \( 70 \).

- For \( \alpha' > 0 \):
  
  Global \( SO(6,6) \) spontaneously broken to \( SU(4) \times SU(4) \).
  
  There are \( 66 - 30 = 36 \) Goldstone bosons. They transform in the \( 6 \otimes 6 \).

- These are type 3) of list we constructed early in the talk:

  3) \( \varphi_s = z \otimes z \quad \text{ex.} \quad \varphi^{12|56} \)

- Eq. (\( \star \)) holds to all orders in \( \alpha' \). \( \text{have checked explicit up to and incl.} \ \alpha'^7 \).
Green, Miller, Russo, and Vanhove (GMRV) have shown that duality and supersymmetry requires the SUSY operator $R^4$ to have a non-linear completion of the form $f_{R^4} R^4$, where $f_{R^4}$ is a moduli-dependent automorphic function which satisfies

$$\Delta f_{R^4} = -42 f_{R^4} \quad \text{for} \quad D = 4$$

Here $\Delta$ is the Laplacian on $E_7(7)/SU(8)$. 
Let’s compare GMRV to our result:

\[
\lim_{\rho_1 \to 0} \left\langle \varphi \varphi + + + - \right\rangle_{R^4} = 2\zeta(3) \frac{6}{5} [34]^4 \langle 56 \rangle^4 \neq 0.
\]

Must come from local operator \((\varphi \varphi)_{\text{sing}} R^4\), so that must be part of the non-linear completion of \(R^4\), i.e. \(f_{R^4} R^4\) with

\[
f_{R^4} \propto -2\zeta(3) \left[ 1 - \frac{6}{5} \left( \varphi^{1234} \varphi^{5678} + 34 \text{ others} \right) + \ldots \right]
\]

The Laplacian on \(E_{7(7)}/SU(8)\) is

\[
\Delta = \left( \frac{\partial}{\partial \varphi^{1234}} \frac{\partial}{\partial \varphi^{5678}} + 34 \text{ inequivalent perms} \right) + \ldots
\]

Indeed we find

\[
\Delta f_{R^4} + 42 f_{R^4} = -2\zeta(3) \left( -\frac{6}{5} \times 35 + 42 \right) + O(\varphi \varphi) = 0 + O(\varphi \varphi)
\]

so our result matches GMRV!
The $R^4$ operator in $D = 4$:

- $\mathcal{N} = 8$ SUSY and $SU(8)$ invariant.
- NOT $E_{7(7)}$ invariant.
- Explains why $R^4$ is not a candidate counterterm . . .
- . . . and why the 3-loop 4-point amplitude is finite.

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban ’07]
Next up: $D^4 R^4$

Closed string effective action

$$S_{\text{eff}} = S_{\text{SG}} - 2 \alpha'^3 \zeta(3) e^{-6\phi} R^4 - \alpha'^5 \zeta(5) e^{-10\phi} D^4 R^4$$
$$+ \frac{2}{3} \alpha'^6 \zeta(3)^2 e^{-12\phi} D^6 R^4 - \frac{1}{2} \alpha'^7 \zeta(7) e^{-14\phi} D^8 R^4 + \ldots .$$

$SU(8)$ average procedure gives unique $D^4 R^4$ matrix elements from $\alpha'^5$ of closed string amplitude.

- NOT $E_{7(7)}$ invariant.
- Single soft limit shows SUSY operator is $f_{D^4 R^4} D^4 R^4$ with
  $$f_{D^4 R^4} \propto - \zeta(5) \left[ 1 - \frac{6}{7} \left( \phi^{1234} \phi^{5678} + 34 \text{ others} \right) + \ldots \right]$$
- Satisfies Green et al’s $\Delta f_{D^4 R^4} = -60 f_{D^4 R^4}$
- Conclude: $D^4 R^4$ is not a candidate counterterm.
- $\mathcal{N} = 8$ SG finite at 5-loops.
Closed string effective action

\[ S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3 \zeta(3) e^{-6\phi} R^4 - \alpha'^5 \zeta(5) e^{-10\phi} D^4 R^4 + \frac{2}{3} \alpha'^6 \zeta(3)^2 e^{-12\phi} D^6 R^4 - \frac{1}{2} \alpha'^7 \zeta(7) e^{-14\phi} D^8 R^4 + \ldots . \]

Matrix elements from \( \alpha'^6 \) of closed string amplitude are polluted by pole terms \( R^4 - R^4 \) from \( \alpha'^3 \times \alpha'^3 \).

- We calculate fully \( \mathcal{N} = 8 \) SUSY'ize \( R^4 - R^4 \).
- Extract \( \langle \varphi \bar{\varphi} + + -- \rangle_{R^4 - R^4} \) and subtract it from \( \langle \varphi \bar{\varphi} + + -- \rangle_{e^{-12\phi} D^6 R^4} \).
- \( SU(8) \) average then gives \( \langle \varphi \bar{\varphi} + + -- \rangle_{D^6 R^4} \), which has non-vanishing single soft scalar limit.
- Satisfies Green et al.'s \( \Delta f_{D^6 R^4} = -60 f_{D^6 R^4} - (f_{R^4})^2 \).
  
  The inhom. term is from \( R^4 - R^4 \).
- NOT \( E_{7(7)} \) invariant.
- Conclude: \( D^6 R^4 \) is not a candidate counterterm.
- \( \mathcal{N} = 8 \) SG finite at 6-loops.
$\mathcal{N} = 8$ SUSY and $SU(8)$-invariant candidate counterterm operators.
What do we know about $L \geq 7$ loops?

$\mathcal{N} = 8$ SUSY and $SU(8)$-singlet candidate counterterm operators and $SU(8)$ 70 operators for their single soft scalar limits.

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Multiplicities found using $SU(2, 2|8)$.

[Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]

For $n > 4$ none of the $L = 7$ operators respect $E_{7(7)}$ compatible. This means that the 4-graviton amplitude determines whether theory finite or not at $L = 7$. 
SUSY, $SU(8)$, $E_{7(7)} \implies \mathcal{N} = 8$ supergravity in 4d finite up to 7-loop order.

First divergence at $L = 7$?

Candidate full superspace integral — but does it vanish?

First divergence at $L = 8$?

Candidate full superspace integral available [Kallosh (1981), Howe & Lindstrom (1981)]