Generating tree amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG

Henriette Elvang (IAS)

Rutgers, Sept 30, 2008

- arXiv:0808.1720 w/ Michael Kiermaier and Dan Freedman
- arXiv:0805.0757 w/ Massimo Bianchi and Dan Freedman
- arXiv:0710.1270 w/ Dan Freedman
1. Motivation

Is $\mathcal{N} = 8$ supergravity perturbatively finite?

*Explicit calculations of loop amplitudes:*
Use generalized unitarity cuts [Bern, Dixon, Kosower, ...] to construct loop amplitudes from products of on-shell tree amplitudes.

*Example:*

$\sum_{\text{intermediate states}} A_{n_1}^{\text{tree}} \times A_{n_2}^{\text{tree}}$

*Our work* focuses on developing efficient calculational methods for explicit construction of any on-shell $n$-point tree amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory and $\mathcal{N} = 8$ supergravity.

$\rightarrow$ Generating functions.

*Applications* to intermediate state sums in unitarity cuts.
How to calculate on-shell tree level scattering amplitudes

- Feynman rules ← very many, very complicated diagrams

- On-shell recursion relations ← very useful
  Get $n$-point amplitudes from $k$-point amplitudes with $k < n$.

- Generating functions ← very efficient

  *Idea:* all $n$-point tree amplitudes of $\mathcal{N} = 4$ SYM encoded in a set of simple Grassmann functions $Z_{n}^{\text{MHV}}, Z_{n}^{\text{NMHV}}, \ldots, Z_{n}^{\text{MHV}}$:

  $$ A_{n}(X_{1}, X_{2}, \ldots, X_{n}) = D_{X_{1}} D_{X_{2}} \cdots D_{X_{n}} Z_{n} $$

  with differential operators $D_{X_{i}}$ in 1-1 correspondence with the states $X_{i}$.

  **Advantage:** obtain amplitude directly without having to first compute set of lower-point amplitudes.
SUSY $\implies$ helicity violating $n$-gluon amplitudes vanish:

$$A_n(+, +, \ldots, +) = A_n(−, +, \ldots, +) = 0.$$ 

- The \textit{simplest} amplitudes are \textbf{MHV} (maximally helicity violating)
  $\rightarrow$ $n$-gluon amplitude $A_n(−, −, +, \ldots, +)$

\textbf{MHV sector:} amplitudes related to $A_n$ via SUSY Ward identities.

- The \textit{next-to-simplest} amplitudes are \textbf{Next-to-MHV}
  $\rightarrow$ $n$-gluon amplitude $A_n(−, −, −, +, \ldots, +)$

\textbf{NMHV sector:} SUSY related (but much harder to solve SUSY Ward identities).

...
Properties of the generating function

Generating functions developed for MHV, NMHV amplitudes + for anti-MHV and anti-NMHV.

Precise characterization of MHV and NMHV sectors, e.g. $A_6(\lambda_+ \lambda_+ \lambda_+ \lambda_+ \phi \phi)$ is MHV in $\mathcal{N} = 4$ SYM.

Counts distinct processes in each sector:

<table>
<thead>
<tr>
<th></th>
<th>MHV</th>
<th>NMHV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N} = 4$:</td>
<td>15</td>
<td>34</td>
</tr>
<tr>
<td>$\mathcal{N} = 8$:</td>
<td>186</td>
<td>919</td>
</tr>
</tbody>
</table>

counting $\leftrightarrow$ partitions of integers!

Simple relationship $Z_n^{\mathcal{N}=8} \propto Z_n^{\mathcal{N}=4} \times Z_n^{\mathcal{N}=4}$ (MHV) clarifies SUSY and global symmetries in map $[\mathcal{N} = 8] = [\mathcal{N} = 4]_L \otimes [\mathcal{N} = 4]_R$ of states and KLT relations $M_n = \sum (k_n A_n A_n')$.

Evaluation of state sums in unitarity cuts of loop amplitudes.
Outline

1. Motivation
2. MHV generating functions in $\mathcal{N} = 4$ SYM
3. Intermediate State Spin Sums
4. Recursion relations $\leftrightarrow$ MHV vertex expansion
5. Next-to-MHV generating functions in $\mathcal{N} = 4$ SYM
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG
7. Outlook
I will use spinor helicity formalism:

- If momentum $p_\mu$ null, i.e. $p^2 = 0$, then
  \[
p_{\alpha\dot{\beta}} = p_\mu (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = |p\rangle^{\dot{\alpha}} \langle p|^{\beta}
  \]
  with bra and kets being 2-component commuting spinors which are solutions to the massless Dirac eqn, $p_{\alpha\dot{\beta}}|p\rangle^{\dot{\beta}} = 0$.

- Spinor products $\langle12\rangle \equiv \langle p_1|^{\dot{\alpha}} \langle p_2\rangle^{\dot{\alpha}}$ and $[12] = [p_1^{\alpha}|p_2]_{\alpha}$ are just $\sqrt{|s_{12}|} = \sqrt{|2p_1 \cdot p_2|}$ up to a complex phase.

- Note $[ij] = -[ji]$ and $\langle ij\rangle = -\langle ji\rangle$. 
2. MHV generating function — $\mathcal{N} = 4$ SYM

States $X_i \leftrightarrow$ differential operators $D_{X_i}$

Amplitude $A_n(X_1 X_2 \ldots X_n) = D_{X_1} D_{X_2} \cdots D_{X_n} Z_n$

First need (state $\leftrightarrow$ diff op) correspondence.
$\mathcal{N} = 4$ SYM has $2^4$ massless states:

1+1 gluons $B^-$, $B^+$

4+4 gluini $F_a^-$, $F_a^+$

6 self-dual scalars $B^{ab} = \frac{1}{2} \epsilon^{abcd} B_{cd}$

4 supercharges $\tilde{Q}_a = \epsilon_{\dot{\alpha}} \tilde{Q}_a^{\dot{\alpha}}$ and $Q^a = \tilde{Q}_a^*$ act on annihilation operators:

$$
\begin{align*}
[\tilde{Q}_a, B_+(p)] &= 0, \\
[\tilde{Q}_a, F_+^b(p)] &= \langle \epsilon \, p \rangle \delta^b_a B_+(p), \\
[\tilde{Q}_a, B^{bc}_+(p)] &= \langle \epsilon \, p \rangle \left( \delta^b_a F_+^c(p) - \delta^c_a F_+^b(p) \right), \quad \text{(consistent with crossing sym. and self-duality)} \\
[\tilde{Q}_a, B_{bc}(p)] &= \langle \epsilon \, p \rangle \epsilon_{abcd} F_+^d(p), \\
[\tilde{Q}_a, F_b^-(p)] &= \langle \epsilon \, p \rangle B_{ab}(p), \\
[\tilde{Q}_a, B^-(p)] &= -\langle \epsilon \, p \rangle F_a^-(p)
\end{align*}
$$
Introduce auxiliary Grassman variable $\eta_{ia}$

$i$ momentum label $p_i$, $a = 1, \ldots, 4$ is $SU(4)$ index.

Associate to each state Grassman diff ops $\partial_i^a = \frac{\partial}{\partial \eta_{ia}}$:

- $B_+(p_i) \leftrightarrow 1$
- $F_+^a(p_i) \leftrightarrow \partial_i^a$
- $B_+^{ab}(p_i) \leftrightarrow \partial_i^a \partial_i^b$
- $F_a^-(p_i) \leftrightarrow -\frac{1}{3!} \epsilon_{abcd} \partial_i^b \partial_i^c \partial_i^d$
- $B^-(p_i) \leftrightarrow \partial_i^1 \partial_i^2 \partial_i^3 \partial_i^4$

This is our (state ↔ diff op) correspondence.
SUSY generators $\tilde{Q}_a = \sum_{i=1}^{n} \langle \epsilon i \rangle \eta_{ia}$ and $Q^a = \sum_{i=1}^{n} [i \epsilon] \frac{\partial}{\partial \eta_{ia}}$ give correct SUSY algebra

$$[Q^a, \tilde{Q}_b] = \delta^a_b \sum_{i=1}^{n} [\epsilon_1 i] \langle i \epsilon_2 \rangle = \delta^a_b \sum_{i=1}^{n} \epsilon_1^\alpha p_{i \alpha \beta} \tilde{\epsilon}_2^\beta \to 0 \quad \text{(mom. cons.),}$$

and

$$[\tilde{Q}, \text{diff op}] = \langle \epsilon p \rangle (\text{diff op})'$$

produces correct algebra on states.
The **MHV generating function** is

\[
Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-, 2^-, 3^+, \ldots, n^+)}{\langle 12 \rangle^4} \delta^{(8)}(\sum_i |i\rangle \eta_{ia}) ,
\]

where

\[
\delta^{(8)}(\sum_i |i\rangle \eta_{ia}) = 2^{-4} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle ij \rangle \eta_{ia} \eta_{ja} .
\]

(The $\delta$-function of Grassman variables $\theta_a$ is $\prod \theta_a$)

- $\eta_{ia}$ — auxiliary Grassman variables
- $a = 1, 2, 3, 4$ — $SU(4)$ indices
- $i, j = 1, 2, \ldots, n$ — momentum labels

**Claim:** any 8th order derivative operator built from (state ↔ diff op) correspondence gives an MHV amplitude when applied to $Z_n^{\mathcal{N}=4}$:

\[
A_n^{\text{MHV}}(X_1, \ldots, X_n) = D_{X_1} \cdots D_{X_n} Z_n^{\mathcal{N}=4}.
\]

Let’s prove this!
Proof:

\[ Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-, 2^-, 3^+, \ldots, n^+)}{\langle 12 \rangle^4} \delta^{(8)}(\sum_i |i\rangle \eta_{ia}) \]

- \( Z_n^{\mathcal{N}=4} \) reproduces pure MHV gluon amplitude \( A_n(1^-, 2^-, 3^+, \ldots, n^+) \) correctly:

\[
\begin{align*}
(\partial_1^1 \partial_1^2 \partial_1^3 \partial_1^4)(\partial_2^1 \partial_2^2 \partial_2^3 \partial_2^4) \delta^{(8)}(\sum_i |i\rangle \eta_{ia}) \\
= (\partial_1^1 \partial_1^2 \partial_1^3 \partial_1^4)(\partial_2^1 \partial_2^2 \partial_2^3 \partial_2^4) (2^{-4} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle i j \rangle \eta_{ia} \eta_{ja}) \\
= \langle 12 \rangle^4.
\end{align*}
\]
Proof:

\[ Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-, 2^-, 3^+ , \ldots , n^+)}{\langle 12 \rangle^4} \delta(8) \left( \sum_i |i\rangle \eta_{ia} \right) \]

- \(Z_n^{\mathcal{N}=4}\) reproduces pure MHV gluon amplitude \(A_n(1^-, 2^-, 3^+, \ldots, n^+)\) correctly:

\[
\begin{align*}
& (\partial_1^1 \partial_2^2 \partial_3^3 \partial_4^4)(\partial_1^1 \partial_2^2 \partial_3^3 \partial_2^4) \delta(8) \left( \sum_i |i\rangle \eta_{ia} \right) \\
& = (\partial_1^1 \partial_2^2 \partial_3^3 \partial_4^4)(\partial_1^1 \partial_2^2 \partial_3^3 \partial_2^4) \left( 2^{-4} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle i j \rangle \eta_{ia} \eta_{ja} \right) \\
& = \langle 12 \rangle^4.
\end{align*}
\]

- \(\tilde{Q}_a Z_n^{\mathcal{N}=4}\) satisfies the MHV SUSY Ward identities:

\[
\begin{align*}
\tilde{Q}_a Z_n^{\mathcal{N}=4} & \propto \left( \sum_{i=1}^{n} |i\rangle \eta_{ia} \right) \delta(8) \left( \sum_i |i\rangle \eta_{ia} \right) = 0.
\end{align*}
\]
Proof:

\[ Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\ldots,n^+)}{\langle 12 \rangle^4} \delta^{(8)}(\sum_i |i\rangle \eta_{ia}) \]

- \( Z_n^{\mathcal{N}=4} \) reproduces pure MHV gluon amplitude \( A_n(1^-,2^-,3^+,\ldots,n^+) \) correctly:

\[
\begin{align*}
(\partial_1^1 \partial_1^2 \partial_1^3 \partial_1^4)(\partial_2^1 \partial_2^2 \partial_2^3 \partial_2^4) \delta^{(8)}(\sum_i |i\rangle \eta_{ia}) \\
= (\partial_1^1 \partial_1^2 \partial_1^3 \partial_1^4)(\partial_2^1 \partial_2^2 \partial_2^3 \partial_2^4) (2^{-4} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle ij \rangle \eta_{ia} \eta_{ja}) \\
= \langle 12 \rangle^4.
\end{align*}
\]

- \( \tilde{Q}_a Z_n^{\mathcal{N}=4} \propto (\sum_{i=1}^{n} |i\rangle \eta_{ia}) \delta^{(8)}(\sum_i |i\rangle \eta_{ia}) = 0. \)

- \( [\tilde{Q}_a, D^{(9)}] Z_n^{\mathcal{N}=4} = 0 \)

encode the MHV SUSY Ward identities:

\[
0 = [\tilde{Q}_a, D^{(9)}] Z_n^{\mathcal{N}=4} = \sum_t D_{X_t} \cdots [\tilde{Q}_a, D_{X_t}] \cdots D_{X_n} Z_n^{\mathcal{N}=4},
\]

\[
0 = \langle [\tilde{Q}_a, X_1 \ldots X_n] \rangle = \sum_t \langle X_1 \ldots [\tilde{Q}_a, X_t] \ldots X_n \rangle.
\]
Proof:

\[ Z_n^{\mathcal{N}=4} (\eta_{ia}) = \frac{A_n(1^{-}, 2^{-}, 3^{+}, \ldots, n^{+})}{\langle 12 \rangle^4} \delta^{(8)} \left( \sum_i |i\rangle \eta_{ia} \right) \]

- \( Z_n^{\mathcal{N}=4} \) reproduces pure MHV gluon amplitude \( A_n(1^{-}, 2^{-}, 3^{+}, \ldots, n^{+}) \) correctly:

\[
\begin{align*}
(\partial_{1}^{1} \partial_{2}^{2} \partial_{1}^{3} \partial_{1}^{4})(\partial_{2}^{1} \partial_{2}^{2} \partial_{2}^{3} \partial_{2}^{4}) \delta^{(8)} \left( \sum_i |i\rangle \eta_{ia} \right) &= \left( \partial_{1}^{1} \partial_{1}^{2} \partial_{1}^{3} \partial_{1}^{4} \right) \left( \partial_{2}^{1} \partial_{2}^{2} \partial_{2}^{3} \partial_{2}^{4} \right) (2^{-4} \prod_{a=1}^{n} \sum_{i,j=1}^{n} \langle ij \rangle \eta_{ia} \eta_{ja}) \\
&= \langle 12 \rangle^4.
\end{align*}
\]

- \( \tilde{Q}_a Z_n^{\mathcal{N}=4} \propto \left( \sum_{i=1}^{n} |i\rangle \eta_{ia} \right) \delta^{(8)} \left( \sum_i |i\rangle \eta_{ia} \right) = 0. \]

- \( [\tilde{Q}_a, D^{(9)}] Z_n^{\mathcal{N}=4} = 0 \)

encode the MHV SUSY Ward identities:

\[
0 = [\tilde{Q}_a, D^{(9)}] Z_n^{\mathcal{N}=4} = \sum_t D X_1 \cdots [\tilde{Q}_a, D X_t] \cdots D X_n Z_n^{\mathcal{N}=4},
\]

\[
0 = \langle [\tilde{Q}_a, X_1 \cdots X_n] \rangle = \sum_t \langle X_1 \cdots [\tilde{Q}_a, X_t] \cdots X_n \rangle.
\]

- MHV SUSY Ward identities have unique solutions.
Proof:

\[ Z_{n}^{N=4}(\eta_{ia}) = \frac{A_{n}(1^{-}, 2^{-}, 3^{+}, \ldots, n^{+})}{\langle 12 \rangle^{4}} \delta^{(8)}(\sum_{i} \langle i \rangle \eta_{ia}) \]

- \( Z_{n}^{N=4} \) reproduces pure MHV gluon amplitude \( A_{n}(1^{-}, 2^{-}, 3^{+}, \ldots, n^{+}) \) correctly:

\[
\begin{align*}
(\partial_{1}^{1} \partial_{1}^{2} \partial_{1}^{3} \partial_{1}^{4}) (\partial_{2}^{1} \partial_{2}^{2} \partial_{2}^{3} \partial_{2}^{4}) \delta^{(8)}(\sum_{i} \langle i \rangle \eta_{ia}) \\
= (\partial_{1}^{1} \partial_{1}^{2} \partial_{1}^{3} \partial_{1}^{4}) (\partial_{2}^{1} \partial_{2}^{2} \partial_{2}^{3} \partial_{2}^{4}) \left(2^{-4} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle i j \rangle \eta_{ia} \eta_{ja} \right) \\
= \langle 12 \rangle^{4}.
\end{align*}
\]

- \( \tilde{Q}_{a} Z_{n}^{N=4} \propto \left( \sum_{i=1}^{n} \langle i \rangle \eta_{ia} \right) \delta^{(8)}(\sum_{i} \langle i \rangle \eta_{ia}) = 0. \)

- \( [\tilde{Q}_{a}, D^{(9)}] Z_{n}^{N=4} = 0 \)

encode the MHV SUSY Ward identities:

\[
0 = [\tilde{Q}_{a}, D^{(9)}] Z_{n}^{N=4} = \sum_{t} D_{X_{1}} \cdots [\tilde{Q}_{a}, D_{X_{t}}] \cdots D_{X_{n}} Z_{n}^{N=4},
\]

\[
0 = \langle [\tilde{Q}_{a}, X_{1} \cdots X_{n}] \rangle = \sum_{t} \langle X_{1} \cdots [\tilde{Q}_{a}, X_{t}] \cdots X_{n} \rangle.
\]

- MHV SUSY Ward identities have unique solutions.

\( \Rightarrow Z_{n}^{N=4} \) produces all MHV amplitudes correctly.
Characterizing amplitudes in the MHV sector of $\mathcal{N} = 4$ SYM:

$$D^{(8)} Z_{n=4} = \text{MHV amplitude}$$

hence

$$\# \text{ MHV amplitudes} = \# \text{ partitions of 8 with } n_{\text{max}} = 4.$$  

MHV amplitudes:

$$
\begin{align*}
8 &= 4 + 4 \quad \leftrightarrow \quad \langle B^- B^- B_+ \ldots B_+ \rangle \\
   &= 4 + 3 + 1 \quad \leftrightarrow \quad \langle B^- F^-_a F^a_+ B_+ \ldots B_+ \rangle \\
   &\quad \quad \ldots \\
   &= 1 + \ldots + 1 \quad \leftrightarrow \quad \langle F^{a_1}_+ \ldots F^{a_8}_+ B_+ \ldots B_+ \rangle 
\end{align*}
$$

Total of 15 MHV amplitudes in $\mathcal{N} = 4$ SYM.
Example:

Calculate \( \langle B^- (p_1) F^1_+ (p_2) F^2_+ (p_3) F^3_+ (p_4) F^4_+ (p_5) B^+ (p_6) \rangle \)

\[
(\partial^1_1 \partial^2_1 \partial^3_1 \partial^4_1)(\partial^1_2)(\partial^3_2)(\partial^4_3)(\partial^4_5) \delta^{(8)} \left( \sum_{i} |i\rangle \eta_{ia} \right)
\]

\[
= (\partial^1_1 \partial^1_2)(\partial^2_1 \partial^2_3)(\partial^3_1 \partial^3_4)(\partial^4_5) \delta^{(8)} \left( \sum_{i} |i\rangle \eta_{ia} \right)
\]

\[
= \langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 15 \rangle
\]

using \( \delta^{(8)} \left( \sum_{i} |i\rangle \eta_{ia} \right) = \left( 2^{-4} \prod_{a=1}^{4} \sum_{i,j=1}^{n} \langle i j \rangle \eta_{ia} \eta_{ja} \right) \),

so

\[
\langle B^- (p_1) F^1_+ (p_2) F^2_+ (p_3) F^3_+ (p_4) F^4_+ (p_5) B^+ (p_6) \rangle
\]

\[
= \frac{\langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 15 \rangle}{\langle 12 \rangle^4} A_n(1^-, 2^-, 3^+, 4^+, 5^+, 6^+).
\]
Outline

1. Motivation
2. MHV generating functions in $\mathcal{N} = 4$ SYM
3. Intermediate State Spin Sums
4. Recursion relations $\leftrightarrow$ MHV vertex expansion
5. Next-to-MHV generating functions in $\mathcal{N} = 4$ SYM
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG
7. Outlook
Example: One-loop MHV amplitude

\[ D_{l_1}^{(4)} \left[ \delta^{(8)}(I) \delta^{(8)}(J) \right] \]

\( D_{l_1} \) and \( D_{l_2} \) distribute themselves between \( \delta^{(8)}(I) \) and \( \delta^{(8)}(J) \). This automatically takes care of the intermediate state sum.
How to evaluate the spin sum: 

\[ D^{(4)}_{l_1} \, D^{(4)}_{l_2} \left[ \delta^{(8)}(I_a) \, \delta^{(8)}(J_a) \right] \]

\[
I_a = |l_1\rangle \eta_{1a} - |l_2\rangle \eta_{2a} + \sum_{\text{ext}} i |i\rangle \eta_{ia}
\[
J_a = -|l_1\rangle \eta_{1a} + |l_2\rangle \eta_{2a} + \sum_{\text{ext}} j |j\rangle \eta_{ja}
\]

Use \( \delta \)-function identity \( \delta^{(8)}(I_a) \, \delta^{(8)}(J_a) = \delta^{(8)}(I_a + J_a) \, \delta^{(8)}(J_a) \) and note that

- \( \delta^{(8)}(I_a + J_a) = \delta^{(8)}(\text{ext}) \) is independent of loop momenta.
- \( \delta^{(8)}(J_a) = 2^{-4} \prod_{a=1}^{4} \sum_{j,j' \in J} \langle jj' \rangle \eta_{ja} \eta_{j'a} = \prod_{a=1}^{4} (\langle l_1 l_2 \rangle \eta_{1a} \eta_{2a} + \ldots) \).

So

\[
D^{(4)}_{l_1} \, D^{(4)}_{l_2} \left[ \delta^{(8)}(I_a) \, \delta^{(8)}(J_a) \right] = \delta^{(8)}(\text{ext}) \, D^{(4)}_{l_1} \, D^{(4)}_{l_2} \, \delta^{(8)}(J_a) = \delta^{(8)}(\text{ext}) \, \langle l_1 l_2 \rangle^4 .
\]

Include prefactors and you have a *generating function* for the cut amplitude!
3. Intermediate state sum

**Example:** One-loop MHV amplitude

\[
D_{l_1}^{(4)} \ D_{l_2}^{(4)} \left[ \delta^{(8)}(I) \ \delta^{(8)}(J) \right]
\]

\(D_{l_1}\) and \(D_{l_2}\) distribute themselves between \(\delta^{(8)}(I)\) and \(\delta^{(8)}(J)\). This automatically takes care of the intermediate state sum.

Have done 1-, 2-, 3-, and 4-loop state sums involving MHV, NMHV, \(\text{MHV}\), and \(\text{NMHV}\) generating functions in \(\mathcal{N} = 4\).
Outline

1. Motivation
2. MHV generating functions in $\mathcal{N} = 4$ SYM
3. Intermediate State Spin Sums
4. Recursion relations $\leftrightarrow$ MHV vertex expansion
5. Next-to-MHV generating functions in $\mathcal{N} = 4$ SYM
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG
7. Outlook
Recursion relations ↔ MHV vertex expansion

- **Recursion relations**: express on-shell \( n \)-point amplitude in terms of \( k \)-point on-shell sub-amplitudes with \( k < n \).

- Even better if sub-amplitudes are MHV
  \( \rightarrow \) **MHV vertex expansion**.

For gluons:

For general \( \mathcal{N} = 4 \) external state:
[Bianchi, Freedman, HE (May 2008)]
[Freedman, Kiermaier, HE (Aug 2008)]

[Cheung (2008)] \( \langle - , \text{anything} \rangle \)-shift OK
[Brandhuber, Heslop, Travaglini (2008)]
[Drummond, Henn (2008)]
3-line shift recursion relations

- Analytically continue amplitudes to complex values by shifts of 3 external momenta:

  \[ p_i^{\mu} \rightarrow \hat{p}_i^{\mu} = p_i^{\mu} + z q_i^{\mu}, \quad \text{for } i = 1, 2, 3. \]

  where

  \[ q_1^{\mu} + q_2^{\mu} + q_3^{\mu} = 0 \quad \leftrightarrow \quad \text{momentum conservation} \]

  \[ q_i^2 = 0 = q_i \cdot p_i \quad \leftrightarrow \quad \text{on-shell} \quad \hat{p}_i^2 = 0. \]

  Achieved by \[ |1] \rightarrow |\hat{1}] = |1] + z\langle 23]|X] \quad (\text{+ cyclic}) \]

  with \[ |X] \] arbitrary “reference spinor”.

- The tree amplitude \[ A_n(z) \] has only simple poles, so if \[ A_n(z) \rightarrow 0 \] for \[ z \rightarrow \infty \], then

  \[ 0 = \oint \frac{A_n(z)}{z} \quad \rightarrow \quad A_n(0) = -\sum_{z \neq 0} \text{Res} \frac{A_n(z)}{z} \]
Result is on-shell recursion relation

\[ A_n(0) = \sum_i A_{n_1} \frac{1}{P_i^2} A_{n_2}, \quad n_1 + n_2 = n + 2 \]

The special 3-line shift ensures that the sub-amplitudes are both MHV if \( A_n \) is NMHV. [Risager (2005)]

\[ \begin{array}{c}
\text{NMHV} \\
\text{MHV}
\end{array} = \sum_i 
\begin{array}{c}
\text{MHV} \\
\text{MHV}
\end{array} \]

→ Now use this to get NMHV gen func.
5. Next-to-MHV generating functions — $\mathcal{N} = 4$ SYM

Consider a single MHV vertex diagram:

Apply MHV gen func to each vertex to derive (details omitted)

$$
\Omega_{n,l}^{\mathcal{N}=4} = \frac{A_{n,l}^{\text{gluons}}}{\langle m_1 P_I \rangle^4 \langle m_2 m_3 \rangle^4} \delta^{(8)}(L_a + R_a) \prod_{a=1}^{4} \langle P_I L_a \rangle
$$

where $L_a = \sum_{i \in L} |i \rangle \eta_{ia}$ and $R_a = \sum_{j \in R} |j \rangle \eta_{ja}$.

[Georgio, Glover and Khoze (2004)]

Each term in $\Omega_{n,l}^{\mathcal{N}=4}$ is order 12 in $\eta_{ia}$’s.

Value of diagram is $D^{(12)} \Omega_{n,l}^{\mathcal{N}=4}$ with $D^{(12)}$ built from the external states.

Sum all diagram gen func’s to get full NMHV gen func:

$$
\Omega_{n}^{\mathcal{N}=4} = \sum_{l} \Omega_{n,l}^{\mathcal{N}=4}
$$
Example:
NMHV gluon amplitude

\[ A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+) = D_1^{(4)} D_2^{(4)} D_3^{(4)} \Omega_n^{\mathcal{N}=4} \]

Partition 12 = 4+4+4.

\( \mathcal{N} = 4 \) SYM:
\[ \# \text{NMHV amplitudes} = \# \text{partitions of 12 with } n_{\text{max}} = 4. \]
Total of 34.
We used MHV vertex expansion from 3-line shift recursion relations, which assumed

$$A_n(z) \to 0 \quad \text{for} \quad z \to \infty.$$ 

Is this OK?
We used MHV vertex expansion from 3-line shift recursion relations, which *assumed* 

\[ A_n(z) \to 0 \quad \text{for} \quad z \to \infty. \]

Is this OK?


— *provided the three lines share a common (upper) SU(4) index.*
In $\mathcal{N} = 4$ SYM, $A_n(\hat{1}, \ldots, \hat{i}, \ldots, \hat{j}, \ldots) \to 0$ for $z \to \infty$ when the 3 shifted states 1, $i$, $j$ share a common (upper) $SU(4)$ index.

Outline of proof:

- Consider first amplitude $A_n$ with state 1 a $-$ve helicity gluon.
- Use [Cheung (2008)]’s result that $[1^-, k]$-shift gives valid BCFW 2-line shift recursion relations

$$A_n = \sum \text{MHV} \quad \text{MHV} + \text{NMHV} \quad \text{MHV}$$

- Perform subsequent $[1, i, j]$-shift: The as $z \to \infty$:
  - diagrams $\text{MHV} \times \text{MHV} \to O\left(\frac{1}{z}\right)$
  - diagrams $\text{NMHV}_{n-1} \times \overline{\text{MHV}}_3 \to O\left(\frac{1}{z}\right)$ using inductive assumption.
- Basis of induction established by careful examination of $n = 6$ cases.
- So $A_n(\hat{1}^-, \ldots, \hat{i}, \ldots, \hat{j}, \ldots) \to 1/z$ for large $z$.
- Use SUSY Ward identities to generalize state 1 to any $\mathcal{N} = 4$ state sharing a common index with $i$ and $j$. 
This proves the validity of the NMHV generating function in $\mathcal{N} = 4$ SYM. It also shows that the MHV vertex expansion is valid for all external states.

Also, the generating function is unique: once established, it does not know which valid 3-line shift it came from!

**Anti-(N)MHV:** The generating function for (N)MHV can be obtained from that of (N)MHV by a Grassman Fourier transform.

We have succesfully applied our generating functions to the evaluation of several 1-, 2-, 3-, and 4-loop intermediate state sums.
Outline

1 Motivation
2 MHV generating functions in $\mathcal{N} = 4$ SYM
3 Intermediate State Spin Sums
4 Recursion relations $\leftrightarrow$ MHV vertex expansion
5 Next-to-MHV generating functions in $\mathcal{N} = 4$ SYM
6 From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG
7 Outlook
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG

- $\mathcal{N} = 8$ SG has $2^8$ massless states:
  1 graviton$^\pm$, 8 gravitinos$^\pm$, 28 gravi-photons$^\pm$, 56 gravi-photinos$^\pm$, 70 self-dual scalars $\phi_{abcd}$.
  Global $SU(8)$ symmetry.

Henriette Elvang (IAS) Generating tree amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG

- $\mathcal{N} = 8$ SG has $2^8$ massless states:
  1 graviton$^\pm$, 8 gravitinos$^\pm$, 28 gravi-photons$^\pm$,
  56 gravi-photinos$^\pm$, 70 self-dual scalars $\phi_{abcd}$.
  Global $SU(8)$ symmetry.

- MHV generating function generalizes directly.
  → Useful for testing map
  $[\mathcal{N} = 4] \times [\mathcal{N} = 4] = [\mathcal{N} = 8]$ at tree level

  → Relationship between global symmetries
  $SU(4) \times SU(4) \leftrightarrow SU(8)$
  included in map and generating functions.
6. From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 8$ SG

- $\mathcal{N} = 8$ SG has $2^8$ massless states:
  1 graviton$^{\pm}$, 8 gravitinos$^{\pm}$, 28 gravi-photons$^{\pm}$,
  56 gravi-photinos$^{\pm}$, 70 self-dual scalars $\phi_{abcd}$.
  Global $SU(8)$ symmetry.

- MHV generating function generalizes directly.
  → Useful for testing map
  \[[\mathcal{N} = 4] \times [\mathcal{N} = 4] = [\mathcal{N} = 8]\]
  at tree level

  → Relationship between global symmetries
  $SU(4) \times SU(4) \leftrightarrow SU(8)$
  included in map and generating functions.

- Natural implementation of NMHV generating function.
  → but it doesn’t work for all possible external states
  of $\mathcal{N} = 8$ SG!
  → because the MHV vertex expansion fails in these cases!
Large $z$ for pure graviton $n$-point amplitude:

$$M_n(\hat{1}^-, \hat{2}^-, \hat{3}^-, 4^+, \ldots, n^+) \rightarrow z^{n-12} \quad \text{for} \quad z \rightarrow \infty$$

Numerically verified for $n = 5, \ldots, 11$.

- When the $M_n(z)$ does not vanish for large $z$ the $O(1)$-term contributes as the residue of the pole at infinity. No (known) amplitude factorization that allows systematic calculation of this part.

- Also “bad” large $z$ behavior for lower point amplitudes, for instance no good 3-line shifts for $\langle \phi^{1234} \phi^{1358} \phi^{1278} \phi^{5678} \phi^{2467} \phi^{3456} \rangle$.

- Intermediate state sums in unitarity cuts of $\mathcal{N} = 8$ SG loop amplitudes performed in terms of $\mathcal{N} = 4$ SYM via the KLT (Kawai-Lewellen-Tye) relations $M_n \sim \sum (k.f.) A_n A_n'$.
7. Outlook

Loops in $\mathcal{N} = 8$ supergravity

Is there a connection between “bad” large $z$ behavior in supergravity tree amplitudes and potential UV divergencies?

Role of $E_{7,7}$?

- 70 scalars of $\mathcal{N} = 8$ SG are Goldstone bosons of spontaneously broken $E_{7,7} \rightarrow SU(8)$.

- How will $E_{7,7}$ reveal itself?
  $\rightarrow$ soft-scalar limits of amplitudes (analogous to soft-pion low-energy theorems of Adler).

- We find that 1-soft-“pion” limits of $\mathcal{N} = 8$ tree amplitudes vanish.

- Note that in pion physics 1-pion soft limits do not necessarily vanish, even in models with pions and nucleons both massless.

- Since our May paper: new results by [Arkani-Hamed, Cachazo, Kaplan (2008)]