Statistics of string vacua

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Abstract

In 1975, Scherk and Schwarz proposed that string theory could unify quantum gravity with all other fundamental interactions. This proposal has met with some success, and its study has led to dramatic theoretical developments with great impact in physics and mathematics.

Despite this success, there is still no direct experimental evidence for or against the theory. How could we hope to get such evidence? Is the theory falsifiable, even in principle? We discuss these questions, and a statistical approach to studying them.

1. Introduction

String/M theory is a prime candidate for a theory unifying all fundamental interaction, including quantum gravity. The theory has passed many impressive consistency tests and suggests answers or at least promising approaches to many questions, such as

- What is the scattering of gravitons and other particles at any energy, below or above the Planck scale?
- What is the origin of black hole entropy?
- Is black hole evaporation unitary? What resolves associated paradoxes?
- What prefers the Standard Model and grand unified gauge groups?
- What is responsible for the large hierarchy between the Planck scale and the electroweak scale?
- What structure leads to family replication?

It even suggests new possibilities which, bizarre as they may seem, appear consistent with present data and could be tested in future experiments, such as "large" extra dimensions of space, and Planck scale modifications of cosmology.

String theory has also led to dramatic progress on other questions of particle physics: it inspired the Seiberg-Witten solution of supersymmetric gauge theory, which led to the first analytic proof of confinement, it has proposed new approaches to QCD phenomenology using both perturbative techniques and the AdS/CFT correspondence, etc., etc.

On a more abstract level, we begin to get some picture of the "stringy geometry" which replaces the differential geometry underlying Einstein's equations, and this has led to dramatic mathematical developments: mirror symmetry, topological field theory, the Seiberg-Witten invariants of four-manifolds, noncommutative instantons, etc., etc.

Despite all these successes, there is still no direct experimental evidence for or against the theory, regarded as a candidate "theory of everything." How could we hope to get such evidence? Is the theory falsifiable, even in principle? Does this matter?

Let us start by explaining the framework in which string/M theorists attempt to make predictions. Along the way, we will contrast this to a generic class of theories of "quantum gravity coupled to matter," for short QGM, based more closely on traditional concepts from quantum field theory.

The starting points for any discussion of fundamental physics have to be the Standard Model of electromagnetic, weak and strong interactions, and general relativity. As with all theories, they attempt to explain or "predict" certain observations and experimental results, in terms of a small number of parameters or other choices.

General relativity is striking in its formal simplicity. It has a single dimensionful parameter, the Newton constant. Fixing this, it is established beyond any doubt to describes gravity at distances up to cosmological scales, but at short distances it has been precisely tested only down to around $100\mu m$.

At cosmological scales, the story is less clear. The prevailing hypothesis at present is that there is a cosmological constant, $\Lambda \sim 10^{-120}$ in the natural units of M_{pl}^4. The data can also be fit by more complicated theories with additional scalar fields, but in any case one needs a tiny dimensionless parameter.

The Standard Model is rather more complicated. There are two subsectors which could be said to have a simplicity comparable to GR, QED with its single dimensionless coupling and the electron mass, and the low energy sector of QCD with a dimensionful scale $\Lambda_{QCD} \sim 100$ MeV and two nearly massless quarks. These sectors have been extremely well tested and form the basis of almost all of real world physics.

The full story involves $SU(3) \times SU(2) \times U(1)$ gauge theory, a moderately complicated spectrum of quarks and leptons, and 19 free parameters. It is presently considered well established at energies up to about 100 GeV, with caveats. Recently, convincing evidence for neutrino mixing has been found. This is not described by the SM, but can be fit within the general framework. Also, the Higgs boson has not been (undisputably) detected.

Looking ahead, there are many arguments that new physics: perhaps supersymmetry, perhaps something else, will be discovered at energies around 1 TeV, to be explored in the coming decade at LHC. Of course, there could be new physics at even higher energies, about which we will only have indirect evidence for many years to come.

Thus, the question of whether a specific theory fits "our universe", in practice changes with time. Physicists in 1930 would have been happy to unify general relativity and electromagnetism, with some theory of the proton. Physicists in 2030 may have a very different point of view from ours.

Anyways, as the experimental sitation stands now, in the sense that we just discussed for the SM and GR, string/M theory is close to being **established** as a valid theory.

As was argued in the mid-1980's, heterotic string compactification on Calabi-Yau manifolds can fairly easily lead to grand unified theories of the class previously postulated as natural extensions of the Standard Model.

The main difficulty not solved at that time was the problem of computing the effective potential. As with generic quantum field theories, the problem of understanding the vacuum structure in string/M theory compactifications can be discussed as that of computing a potential function of all scalar fields, whose minima are candidate "phases" or vacua.

$$\frac{\partial V(g,\lambda,\phi)}{\partial(g,\lambda,\phi)} = 0.$$

$$\frac{\partial V(g, \lambda, \phi)}{\partial(g, \lambda, \phi)} = 0.$$

In string/M theory, all values of the coupling constants we observe are determined from the expectation values of the scalar fields. This is why we denoted some fields as g and λ in the above.

In supersymmetric theories, most of the interesting structure of the effective potential comes from non-perturbative effects, and thus one needs non-perturbative or even exact results.

During the mid-90's, techniques based on ideas such as duality and holomorphy have led to great advances in our ability to compute the effective potential. In a real sense, the problem for general supersymmetric gauge theories is under control. A usable partial understanding is available for the larger supergravity theory produced by string/M compactification.

Within this context, one can exhibit supersymmetry breaking and stabilization of all scalar fields appearing in many classes of compactification (Calabi-Yau, M theory on G_2, F theory).

At this point, although no concrete string/M theory compactification solves all the problems, there is no theoretical aspect of the SM+GR which seems out of reach, except possibly one: the cosmological constant problem.

There is one idea for solving this problem which seems viable, and indeed rather conservative. It is the "statistical" idea, growing out of work of Brown, Teitelboim, Banks, Feng *et al* and especially (Bousso and Polchinski, 2000).

This idea requires for its realization a theory with huge numbers of vacua, at least 10^{120} in the most general (nonsupersymmetric) picture. Furthermore, the cosmological constants in these vacua, each defined as the value of the effective potential at the minimum, must be uniformly distributed near $\Lambda \sim 0$.

$$\Lambda \in \{V(\phi_1), V(\phi_2), V(\phi_3), \ldots\}.$$

Then, it will be statistically likely for vacua to exist with the observed small Λ. Of course, given very strong computational ability, one could go on to check that indeed one or several of these vacua actually do have $V_{min} = \Lambda_{obs}$.

Bousso and Polchinski suggested that the requisite large numbers of vacua could be obtained by varying the choice of flux in the compactification. By flux, we mean the expectation values of the higher p-form field strengths present in supergravity, integrated on non-trivial cycles in the compactification manifold.

In a bit more detail, let F be a gauge field strength; the equations of motion $\nabla F = 0$ force it to be harmonic, so determined by its integral over non-trivial homology cycles Σ_α. Let

$$N^\alpha = \int_{\Sigma_\alpha} F$$

be the quantized number of F fluxes on the cycle Σ_α, and K be the number of cycles.

A qualitative description of the total energy is is

$$E = E_0 + \frac{1}{l^4} \sum_{i=1}^{K} q_i(z)^2 N_i^2$$

where E_0 is a flux-independent contribution, q_i is a "charge" (determined by kinetic terms) and l is the length scale of the internal space.

Suppose $E_0 < 0$ and $q \sim 1$, then the number of flux vacua with given $\Lambda = E(N)$ is roughly

$$d\mu_{vac}(\Lambda) \sim \int d^K N \ \delta(\Lambda - E) \tag{1}$$
$$\sim (\Lambda - (E_0 l^4))^{K/2-1} . \tag{2}$$

Thus, the number of vacua with $\Lambda \sim 0$ is roughly

$$d\mu_{vac}(\Lambda \sim 0) \sim L^{K/2-1}$$

with $L = E_0 l^4$. Since typical Calabi-Yau's have $L \sim 100 - 500$, this motivates estimates such as $N_{vac} \sim 10^{100} - 10^{500}$.

Now this is only a heuristic argument, ignoring questions like back reaction on the metric, duality and so forth. We will come back to this question with a more careful treatment in the later part of the talk. For now, we simply state that there is evidence that string/M theory compactification can produce vacuum multiplicities of the sort we need to solve the cosmological constant problem.

Although this would be a talk in itself, it seems fair to say that existing work made the claim that string/M theory contains vacua which could reproduce the Standard Model plausible, with the chief the chief remaining obstacle to the claim being the cosmological constant problem. Thus the arguments we just gave significantly strengthen the plausibility argument.

Of course, other theories might also meet these standards. Indeed, the hypothetical "QGM" theory might have had no trouble meeting this test. If it were possible to couple **any** renormalizable gauge theory to matter, then since the Standard Model is renormalizable, it would have no difficulty at all. One would simply choose the known matter content and adjustable couplings. Furthermore, in conventional quantum field theory, the vacuum energy is an adjustable parameter, set during renormalization.

The reason one has to go to much more trouble to establish the same claim from string/M theory, is that this theory (presumably) can **not** reproduce every theory of matter coupled to gravity. Not every gauge group and matter content can be realized; not every joint value of the coupling constants is possible. This is what makes the theory potentially more predictive than QGM.

But let us now examine the claim that string/M theory can "predict" the SM and physics yet to be discovered, a little more closely. On reflection, it clearly depends on details of the set of solutions, and especially the number of solutions. Let us contrast some possibilities.

If the theory had only one solution which looked anything like the real world, there would be no problem: either it agreed with the SM at low energies or it did not.

Even if there were 100 solutions, it would be easy to go through the list. Only if several solutions reproduced SM physics, and led to different subsequent predictions, would there be any ambiguity. While possible, this is still far more predictive than most theories, and far more so than the generic QGM.

At the other extreme, there might turn out to be an infinite number of solutions, densely spread through the "space of theories." In this case, it would be very hard to claim that the theory made any prediction, or was falsifiable on this level. No matter what "principle" we came up with to select a solution, if the proposed solution did not agree with the data, we could always claim that the principle was wrong and that a different solution was the correct one.

This is not to say that the theory is not falsifiable. Rather, it would be no more falsifiable than the SM, or the general class of QGM's. There would still be an infinite number of predictions of the same general type made by the SM.

The difficulty of constructing solutions of string/M theory, and our limited nonperturbative understanding of the theory, still make statements like these controversial: it has not been **proven** that there are many, let alone 10^{100}, consistent and sufficiently stable vacua; not even **one** vacuum which reproduces all SM physics has been constructed; perhaps there is a more attractive solution to the cosmological constant problem, and so on.

However, these statements do properly summarize our present state of understanding of the problem, and are a consistent picture of how contact with the real world will eventually be made.

On this general level, they are not special to string/M theory. It might also turn out that a particular QGM theory could not be coupled to an arbitrarily chosen renormalizable matter theory. It might turn out to constrain the specific field content, or even the specific couplings. Again, the value of this will depend on how many different consistent allowed theories or solutions the framework admits.

So, how many solutions would be "too many" from this point of view?

One point for which one can be relatively quantitative is the question of predicting couplings. Suppose one considers the class of models with SM spectrum, but arbitrarily chosen consistent values of the 19 parameters. One can ask, what is the volume in the 19-dimensional parameter space, consistent with present day observation? We define volume using Lebesgue measure, with unit normalization for dimensionless quantities, and to the Planck scale for dimensionful quantities (the Higgs mass).

The volume is about 10^{-120}, with 10^{-40} or so for the Higgs mass, 10^{-10} for each of the fine structure constant, electron mass, and proton mass, and various factors for the other couplings.

If we also factor in the cosmological constant in Planck units, the total volume is about 10^{-240}. Thus, in a uniform distribution of 10^{240} theories, one might expect 1 theory to realize the SM couplings.

Of course, this "statistical" scenario for realizing the SM couplings would not predict the SM at all. Every other comparable volume in parameter space would also be likely to contain a theory. On the other hand, having found "our" theory, we could then go on to make predictions about physics to be discovered later.

On the other hand, if there were many more than 10^{240} vacua, then many would be expected to agree with the SM. If this expectation were borne out by the actual distribution of vacua, and if the different vacua which agreed with the SM made different predictions, then the danger arises that it will be impossible to falsify the theory by making new measurements which disagree with all of the predictions.

Thus, in a rough sense, we could take 10^{240} or so as the number of vacua above which predictivity becomes problematic. This estimate did not include the difficulty of actually realizing the SM matter content. It did not account for supersymmetry, which can solve the hierarchy problem, bringing the 10^{240} down to 10^{120}. Finally, it made the simplistic assumption that vacua are uniformly distributed in parameter space. To discuss these points better, one must define "vacuum" more precisely.

Still, it shows the order of number we want: if there are many fewer than 10^{60} vacua (with supersymmetry), we need much better arguments to claim that we can solve the cosmological constant problem, while if there are many more than 10^{240}, we face potential loss of predictivity and testability beyond that of the SM or a general QGM framework.

It is interesting that present estimates of the number of vacua are in the 10^{100} range, completely independently of the arguments we just gave or any desired application. The best understood case is Calabi-Yau compactification.

First, the number of distinct CY threefolds is believed to be 10^5–10^6. This has not been proven and for all mathematicians know, the number might turn out to be infinite. Still, a fair amount work has been done on constructing CY's leading to this belief.

One then must make additional choices of bundle (heterotic string), or brane configuration (type I and II strings). In the case of bundles, it is mathematically proven that the number of additional choices is finite, and more generally there are plausibility arguments. We have only loose estimates of these numbers, say 10^{10} per CY or so.

We mentioned the choice of flux, which potentially brings in up to 10^{500}, which would be too many. This is still a rough estimate and work proceeds on pinning this number down. It now appears quite possible that simple additional conditions, such as obtaining acceptable supersymmetry breaking, or requiring a discrete symmetry to forbid fast proton decay, will bring this number down significantly.

Anyways, it may be possible to make a rough but controlled estimate of this number in the near term, assuming of course that there are not huge numbers of yet undiscovered vacua.

Suppose we decide that we expect one "candidate" vacuum out of this large number? How can we find it? Testing the conditions we discussed, especially the small cosmological constant, appears very difficult.

Furthermore, at present there is no reasonable candidate for an *a priori* selection principle, even one which only suggests where to look. Now it is definitely worth looking for such principles. For example, one might try to find a "measure" which weighs different vacua according to how "likely" they are to emerge from early cosmology. Even doing this would require fairly detailed knowledge about the set of possible vacua, and the configuration space which contains the vacua. Most such ideas require even more information. It does not seem reasonable to hope for a principle which will tell one in advance which string theory, Calabi-Yau, brane configuration etc. to look at.

And, we should keep in mind, that there is no guarantee that any *a priori* vacuum selection principle exists. We only have one sample, and the question of why we observe this one need not have any better answer than "because we are here."

In our opinion, the primary question string theorists, or QGM theorists, face, is whether there exists a solution of the theory which reproduces our data and could make predictions. One does **NOT** need anthropic arguments to justify considering a particular solution which matches our observations, any more than fixing the matter content and parameters of the Standard Model requires an anthropic argument.

There are meaningful anthropic arguments, which are more specific and predictive type of argument. Depending on the set of possibilities, one might find these interesting, or not, depending on taste. It seems premature to go too far in this direction before the primary question is answered.

2. Statistics of vacua

So what to do? To properly address any of the questions we raised, and know whether string theory is predictive under a certain set of assumptions, we need some ability to work with and make tests on "all" the vacua.

While we are not yet able to do this in an exact way, we may be reaching the point where we can do it in a useful approximate way.

To do this, we **grant** that the set of "all" relevant vacua, under some precise definition, exists. Our working definition of "vacuum" is provided by effective field theory: we consider every low energy four dimensional effective field theory which could be derived from string/M theory, and consider every (sufficiently metastable) minimum of the effective potential, up to possible equivalences under duality.

While we should work to prove that this is a precise definition within string theory, it fits with present intuitions, and of course the Standard Model we are trying to make contact with **is** an effective field theory.

We then make a precise hypothesis for an approximate description of this set: they appear in a specific ensemble of effective field theories. While the ensemble should be precisely specified, we need not claim that it exactly represents the set of string/M theory vacua, only that it represents it well enough for our purposes.

We then can proceed in two directions:

- We can test whether our hypothesized ensemble is accurate, by comparing with actual string/M theory constructions.
- We can find out what fraction of vacua out of our ensemble meet a specified phenomenological test.

Let us give a very simple example to illustrate the point, by asking the question:

How many string/M theory vacua have $SU(3) \times SU(2) \times U(1)$ gauge symmetry unbroken at low energy?

If we define our terms, and if string/M theory has a precise definition, **and** if there are finitely many physically distinct vacua, then this question has a definite answer.

One can just as easily generalize the question to, out of all vacua, how many have low energy gauge group G ? Let us denote this number by the function

$$d\mu[G].$$

While finding this function exactly is hard, perhaps it can be approximated in some simple and useful way.

For example, could it be that the rank $r = \mathrm{rk}\, G$ of the unbroken gauge group, roughly satisfies a power law distribution,

$$d\mu[r] \sim N \times r^{-\alpha}.$$

If so, and if we could estimate N and α, we could get a rough estimate for how many vacua have a rank 4 gauge group, without much effort. One could go on to study the distribution $d\mu[N_1, N_2, \ldots]$ of the ranks of the simple factors, etc.

Another question we can ask, is the number of theories with a given matter content. In (Douglas 0303194), this question was considered for quiver gauge theories arising from type II string on Calabi-Yau manifolds. and arguments were given that the fraction of gauge theories with a given matter content is usefully approximated by the distribution

$$d\mu[I_{ij}] \sim \prod_{i<j} \frac{dI_{ij}}{|I_{ij}|}.$$

Since this is a precise ensemble of theories, any specific matter content, for example that of the Standard Model, appears as some definite fraction of the models. If we consider models from this ensemble with the correct gauge group $U(3) \times U(2) \times U(1)^2$, the correct matter content of the brane construction we just described, appears in a fraction 3×10^{-6} of models.

Thus, we have formulated a precise and quantitative sense in which the Standard Model matter content is "generic" in this ensemble and construction.

While this ensemble is a bit oversimplified, a description of the true ensemble of brane gauge theories, which gives a good estimate for the fraction of models which work, might not be too much more complicated. One can refine our estimate by formulating more detailed ensembles, and comparing them with actual string theory constructions. We suspect this will lead to similar results, say

$$10^{-16} < \frac{N_{SM}}{N_{\text{all }G,R}} < 1.$$

If so, then realizing the Standard Model spectrum is not the hard part of the problem.

3. Flux vacua

Let us come back to the problem of counting **flux vacua** obtained by compactifying type II or heterotic strings on a Calabi-Yau threefold, with p-form gauge field strengths. Their contributions to the energy depend on Calabi-Yau moduli, so this leads to a potential which can stabilize moduli and break supersymmetry. (Strominger; Polchinski; K. and M. Becker; Giddings; Kachru; Trivedi; many others ...)

Furthermore, by known dualities (Maldacena, Gopakumar, Vafa, Klebanov, Strassler, ...), these effective potentials describe a good deal of nonperturbative physics, such as gauge theory instanton effects. It is not completely crazy to claim that most of the choices not having to do with explicit low energy gauge symmetry can be dualized into this choice. If so, this multiplicity would be the dominant factor in the total multiplicity of vacua.

In any case, these vacua, supersymmetric and nonsupersymmetric, form a well defined "ensemble," and thus we can study their statistics:

- How many are there ?
- How are they distributed in moduli space ?
- How are cosmological constant and supersymmetry breaking scales distributed ?

In the remainder of the talk, we will discuss how to get approximate but **controlled** results for these questions, using techniques which generalize to a wide class of similar problems.

The effective flux potential can be computed exactly at large volume, using special geometry and the superpotential (Gukov, Vafa, Witten):

$$
\begin{aligned}
W(z) &= \int_M (F^{(3)}_{RR} + \tau H^{(3)}_{NS}) \wedge \Omega(z) \equiv \int_M G \wedge \Omega(z); \\
K(z,\bar{z}) &= -\log \int_M \Omega(z) \wedge \bar{\Omega}(\bar{z}); \\
V(z) &= e^K \left(|DW|^2 - 3|W|^2 \right).
\end{aligned}
$$

Here z parameterizes complex structure moduli of M, $\tau = C^{(0)} + ie^{-D}$ is the axion-dilaton, and $\Omega(z)$ is the holomorphic three-form on M. We will discuss Kähler moduli, whose potential is determined non-perturbatively, later.

This formula is appropriate for the type IIb string. The heterotic string leads to something similar with one type of flux, not two.

Let us proceed to count supersymmetric type II flux vacua. Essentially the same considerations apply to F theory. Define

$$N^{\alpha} \equiv N^{\alpha}_{RR} + \tau N^{\alpha}_{NS} = \int_{\Sigma_\alpha} F^{(3)}_{RR} + \tau H^{(3)}_{NS},$$

so that

$$W_N(z) = \sum N^{\alpha}\Pi_{\alpha}(z)$$

in terms of the periods Π of the holomorphic three-form. These are explicitly computable (complicated) functions, which satisfy the relations of "special geometry."

A supersymmetric vacuum is then a particular solution of

$$D_i W_N(z) = (\frac{\partial}{\partial z_i} + \frac{\partial K}{\partial z_i})W = 0.$$

These vacua are related by dualities, which act both on flux and moduli,

$$(N, z) \sim (N', z').$$

We can take one representative from each duality class, by only considering vacua for which the moduli z to live within a fundamental region of the duality group. On doing this, N^{α}_{RR} and N^{α}_{NS} can be arbitrary integers, so the first point is to understand why the total number of allowed choices is finite.

In principle, we might have to make cuts to get finiteness. Two physically reasonable ones are to remove decompactification limits (the large complex structure limit), or to put a cut on the cosmological constant, $\Lambda = -3e^K|W|^2 \geq -|\Lambda_{min}|$. For simplicity, we won't do this unless forced to.

Other considerations can also cut down the possible choices of flux. An intriguing aspect of the IIb problem is that the IIb Chern-Simons coupling

$$\int C^{(4)} \wedge F^{(3)} \wedge H^{(3)}$$

leads to the following constraint from tadpole cancellation for $C^{(4)}$ (Gukov *et.al.*, Giddings *et.al.*):

$$\int F \wedge H + N(D3 \text{ branes}) = N(O3 \text{ planes}), \tag{3}$$

The numbers $N(O3)$, $N(D3)$ are positive for supersymmetric vacua. Furthermore, one can show that

$$0 < \int G \wedge *G \propto \int F \wedge H$$

for supersymmetric vacua. This might suggest that the number of fluxes satisfying (3) should be finite.

However, this is not true, because $\int F \wedge H$ involves the intersection form,

$$\int F \wedge H = \sum_i N_{RR}^{Ai} N_{NS}^{Bi} - N_{RR}^{Bi} N_{NS}^{Ai},$$

which is an **indefinite** form. Thus,

$$0 < \int F \wedge H \leq L_{max}$$

has an infinite number of solutions.

In fact, (Trivedi & Tripathy 0301139) have found infinite series of supersymmetric vacua on $K3 \times T^2$ (ignoring dependence on Kähler moduli).

So, finiteness of the number of vacua was not established. On the other hand, the series found by T & T decompactifies. So, putting a "cut" which removes decompactification limits saves it, in this example.

One can show (Ashok and Douglas 0307049) that all infinite series of IIb supersymmetric flux vacua, run off to limits of moduli space, in which the conditions $D_i W_N = 0$ change rank. The only known examples are decompactification (large complex structure) limits. So, this cut should suffice.

4. Counting vacua

Our basic approximation is to replace the sum over quantized fluxes by an integral, which can be justified for large L. In the spirit of the introduction, this will give us a distribution of "likely" vacua, not in a true probabilistic sense, but simply as a reflection of our approximate treatment.

We then treat this integral as a limit of a Gaussian ensemble of superpotentials, defined as a distribution

$$d\mu[W] = \int d^{2K}N\ e^{-Q_{\alpha\beta}N^\alpha \bar{N}^\beta}\delta(W(z) - \sum_\alpha N^\alpha \Pi_\alpha(z))$$

with

$$\Pi_\alpha(z) = \int_{\Sigma_\alpha} \Omega(z).$$

Any expectation value in this ensemble can be computed in terms of a two-point function,

$$\left\langle W(z_1)W^*(\bar{z}_2)\right\rangle,$$

which gives the joint expectation value for the product of superpotentials at two points in moduli space.

The natural two-point function in the IIb flux problem is

$$\begin{aligned}
\left\langle W(z_1)W^*(\bar{z}_2)\right\rangle &= \sum_G e^{-\int \alpha G\wedge * G}\, W(z_1)\bar{W}(\bar{z}_2) \\
&\sim \int dG\; e^{-\int \alpha G\wedge * G}(G\wedge\Omega(z_1))\;(\bar{G}\wedge\bar{\Omega}(\bar{z}_2)) \\
&= -\frac{1}{\alpha}\int \Omega(z_1)\wedge\bar{\Omega}(\bar{z}_2), \\
&= \frac{1}{\alpha}e^{-K(z_1,\bar{z}_2)},
\end{aligned}$$

(using a standard formula from special geometry), where $K(z_1, \bar{z}_2)$ is the Kähler potential, regarded as an independent function of the holomorphic and antiholomorphic variables.

This respects the $Sp(b_3, \mathbb{Z})$ group of possible duality symmetries. It also allows us to fix the tadpole condition $L = \eta NN$, by doing a Laplace transform,

$$\left\langle \ldots \right\rangle_{\text{fixed } L} = \int d\alpha\; e^{\alpha L}\int d^K N\; e^{-\alpha G\wedge * G}\ldots . \int d\alpha\; e^{\alpha L}\left\langle \ldots \right\rangle_{\text{fixed } \alpha}.$$

(A subtle point is that this "action" is not positive definite. One can however justify its use, by an analytic continuation. This works because $\int F\wedge H > 0$ for susy vacua.)

Using

$$\left\langle W(z_1)W^*(\bar{z}_2)\right\rangle = e^{-K(z_1,\bar{z}_2)},$$

one finds that the number of any given type of vacuum, can be computed as the integral of a density constructed from the Kähler form ω and curvature R of moduli space. These can be determined using techniques developed in the study of mirror symmetry.

The simplest computations are of analytic expectation values, which can be computed using Wick's theorem. In particular, the continuous flux approximation to the "supergravity index," which counts vacua with signs (the sign of the determinant of the fermion mass matrix),

$$\begin{aligned} I_{vac} &= \sum_{vacua} (-1)^F \\ &= \int_{\mathcal{F}} d^n z \, \left\langle \delta(DW(z)) \ \det D^2 W(z) \right\rangle, \end{aligned}$$

as an integral of an "index density"

$$d\mu_I(z) = \left\langle \delta(DW(z)) \ \det D^2 W(z) \right\rangle$$

counting the contribution of supersymmetric vacua which stabilize the moduli at the point z.

Computing this, simply requires computing quantities such as

$$\left\langle D_{1a} D_{1b} W(z_1) \bar{D}_{2\bar{c}} \bar{D}_{2\bar{d}} W^*(\bar{z}_2) \right\rangle |_{z_1=z_2=z} = R_{a\bar{c}b\bar{d}} + g_{b\bar{c}} g_{a\bar{d}} + g_{a\bar{c}} g_{b\bar{d}},$$

$$\left\langle \bar{D}_{1\bar{a}} D_{1b} W(z_1) D_{2c} \bar{D}_{2\bar{d}} W^*(\bar{z}_2) \right\rangle |_{z_1=z_2=z} = g_{\bar{a}b} g_{c\bar{d}},$$

and taking a determinant.

The result is an index density for the distribution of vacua over moduli space,

$$d\mu_I(z) \propto \det(R + \omega)|_z,$$

where ω and R are the Kähler metric and its curvature.

Heuristically, this result says that vacua are distributed roughly one per flux sector (which we will count below) per **unit volume** in configuration space (measured in units of M_{pl}^{2n}, but curvature can modify this.

Doing the Laplace transform produces the total index of all supersymmetric flux vacua with flux up to L. Essentially, we have now scaled out all the moduli-dependent factors, and the remaining factor is the volume of a $2K$-sphere in flux space:

$$\begin{aligned} I_{vac}(L \leq L_{max}) &= \frac{(2\pi L)^{b_3}}{12 \cdot b_3!}[c_n(\Omega M \otimes \mathcal{L})] \\ &= \frac{(2\pi L)^{b_3}}{12\pi^n b_3!}\int_{\mathcal{F}} \det(-R-\omega), \end{aligned}$$

where $\mathcal{F}$ is a fundamental region in the complex structure moduli space.

For example, for T^6 (with symmetrized period matrix), $K = b_3 = 20$, and

$$I = \frac{1}{1008 \cdot 12 \cdot 20!}(2\pi L)^{20} \sim 2 \times 10^{23} \text{ for } L = 32.$$

The index provides a lower bound for N_{vac}. N_{vac} is also computable, by doing more complicated Gaussian integrals. For example, with one field z, one has

$$\begin{aligned} d\mu_c &= \tfrac{1}{\pi}|R-N| \cdot \omega; & R > 2N, \\ &= \tfrac{(R-2N)^2+N^2}{\pi|R-3N|} \cdot \omega; & R < 2N. \end{aligned}$$

To complete the discussion, we need to discuss Kähler stabilization. Note that the W we considered does not depend on Kähler moduli ρ, leading to no-scale structure at tree level. This is generically spoiled by α' and non-perturbative corrections, say

$$W_{NP} = e^{iN\rho} + \ldots .$$

which can be arranged to appear from some brane world-volume theory. Very generally (KKLT, Douglas), a solution of $DW(z) = 0$ for the complex structure moduli, with $e^K |W|^2 << M_{pl}^4$, will become a stable supersymmetric AdS vacuum once these are taken into account. For example,

$$0 = D_\rho W = iNe^{iN\rho} - \frac{3}{\rho - \bar{\rho}} W_{rest}.$$

has a solution for

$$\frac{2N}{3} (\mathrm{Im}\ \rho) e^{iN\rho} = W_{rest}.$$

The function on the l.h.s. can take any value up to $2/3e \sim 1$, and one expects an exact nonperturbative $W(\rho)$ to behave similarly. Thus, any vacuum with W_{rest} not too large, can be stabilized.

Thus, we need to know the distribution of AdS cosmological constants.

This distribution can be computed in the same way as above (we denote $\Lambda = -3E$ with $E = e^K|W|^2$):

$$\begin{aligned} d\mu(\Lambda) &= \left\langle \delta(DW) \;\det D^2W \;\delta(E - e^K|W|^2) \right\rangle \\ &= \left(\frac{(2\pi)^{b_3}}{12\pi^n}\right) \times \left[\frac{(L-E)^{b_3-1}}{(b_3-1)!} \int_{\mathcal{F}} \det(-R - 2\omega) + \frac{\Lambda(L-E)^{b_3-2}}{(b_3-2)!} \int_{\mathcal{F}} \omega^n\right]. \end{aligned}$$

In particular, the number of vacua with small $\Lambda = \epsilon << L$ goes as $N_{vac} \times \epsilon b_3/L$, so the distribution does not fall off at zero. A simple intuition suggesting this result is the following: the period vector Π^α defines a "direction in flux space," and the constraint of small Λ is simply projecting onto a plane orthogonal to this vector. This produces the volume of a $2K-2$-sphere.

While the claim is correct in this example, this argument assumes that the vector Π^α is not correlated with the other quantities which appear. This is false (for example) in the analogous heterotic string problem, in which the distribution of AdS cosmological constants goes as

$$d\mu(\Lambda) \sim \Lambda^{K/2}.$$

The upshot in the IIb problem is that, taking (say) $\epsilon \sim 10^{-3}$, we obtain a lower bound 10^{20} on the number of flux vacua on T^6. One mighe be even more conservative and cut out the strong coupling regime. If we do this by insisting on $\mathrm{Im}\,\tau > 25$, we find that $N_{vac}/25 \sim 10^{19}$ vacua satisfy this constraint.

Thus it seems hard to get around the conclusion that string theory has many vacua. CY's are known with $b_3 \sim 500$ and $L \sim 10^4$, so there is some danger that the number of vacua is large enough to spoil predictivity. The fraction of vacua with the Standard Model spectrum, which as we argue elsewhere is around $10^{-10\pm 5}$, is not small compared to these numbers. It could still be that some of the other physical constraints (stable susy breaking, inflation, etc.) are so difficult to meet, that the number of viable vacua is ~ 1.

So, let us proceed to count non-supersymmetric vacua.

5. Susy breaking

We define a non-supersymmetric vacuum as a solution of

$$\frac{\partial V(z)}{\partial z^i} = 0$$

with

$$V = e^K \left(g^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3|W|^2 \right) + D^2.$$

It is metastable if $V'' \geq 0$ (*i.e.*, it is tachyon free).

One can again debate the applicability of effective field theory in this case, and the question of whether this formula applies after supersymmetry breaking. While one can always find K for which this holds, one can worry that explicit flux-dependent corrections are more important. Anyways, let us continue.

$$V = e^K \left(g^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3|W|^2 \right) + D^2 .$$

One can see very generally that energy from antibranes and gauge fluxes comes in as the D^2 part of the potential. Thus, the non-supersymmetric vacua discussed by (KKLT, Kallosh *et al*), and others, work by supersymmetry breaking by D-terms. These appear comparable in number to AdS vacua.

On the other hand, it could be, even setting $D = 0$, that the flux potentials $V(z)$ have metastable minima. They are certainly complicated enough functions.

Indeed, there is a simple heuristic argument that says that non-supersymmetric vacua of this type are roughly as common as supersymmetric vacua. The conditions $V' = 0$ are again as many equations as unknowns, and since $V \sim (DW)^2$ is of "higher degree" than W, should have even more solutions.

The tachyon-free condition $V'' \geq 0$ can be studied by finding the distribution of these masses. Typically, massive fields are roughly as common as tachyons, so with $2n$ real fields one expects a fraction 2^{-2n} of vacua to be tachyon free, a fairly large fraction in the present context. So, more precise arguments are interesting.

In (Denef and Douglas, to appear), we get the distribution of non-supersymmetric vacua with a specified susy breaking parameter

$$F_i = D_i W(z).$$

For such vacua,

$$\partial_j V = e^K \left(F^*_{\bar{i}} g^{\bar{i}k} D_j D_k W - 2 F_j W^* \right)$$

and we linearize the condition $V' = 0$. Then, we need

$$d\mu_{ns}(z, F) = \Big\langle \delta(F - D_i W)\ \delta(V')\ \det(DDW, V'') \Big\rangle.$$

One could then integrate over F if desired (in phenomenology, one generally does not).

The simplest cases are when the number of fluxes equals the number of unknowns, as one can then just solve for the flux. In particular, in IIb compactification, there are $2b_3$ real fluxes, and $2b_3$ variables (z^i, F_i) (counting the dilaton-axion).

Thus, one needs to solve for $N(z,F)$ and change variables, to find the distribution.

Somewhat surprisingly, there is a simple universal solution in this case. Using F theory notation (the fluxes are four-forms, and N^α is real), we find

$$N^\alpha = \eta^{\alpha\beta}\left(F_i g^{i\bar{j}} D_{\bar{j}} \Pi_\beta(z) + \text{c.c.}\right).$$

This can be seen to solve both $F_i = D_i W$ and

$$0 = D_i D_j W_N.$$

So, these are formally *de Sitter* nonsupersymmetric vacua with $W=0$ and $V = |DW|^2$.

These are not physical de Sitter vacua – we have not stabilized the Kähler modulus, and these vacua actually run off to large volume. They can be understood in ten-dimensional terms as vacua with *anti-self-dual* flux (where susy were self-dual flux), with the overall volume fixed by hand. Because they start out as de Sitter, the KKLT type of stabilization doesn't work here.

Rather, we have confirmed the intuitive argument that non-supersymmetric vacua are as common as supersymmetric, but with a twist: none of these nonsupersymmetric vacua could be realistic, as all have $\Lambda \sim M^4_{susy}$. If (z, F) satisfy special relations, there can be other non-supersymmetric vacua with $W \neq 0$, but these look rare.

Granting the validity of our approximations, one would conclude that in realistic IIb compactification, pure F term breaking is much harder to accomplish than D term or mixed D-F breaking.

More generally, other flux ensembles do contain nonsupersymmetric F type vacua with $\Lambda \sim 0$; for example the heterotic string. One can then get conditions on the geometry of moduli space which favor or disfavor such vacua (to appear).

6. Summary and conclusions

At present, while it appears that string/M theory could contain vacua which describe our universe, there is no good way to find them. Furthermore, we do not know whether we should expect to find one or many such vacua, and what type of predictions string/M theory should make.

We discussed a statistical approach to deal with this problem, which proceeds as follows:

- We propose a precise ensemble of effective field theories which is supposed to model the set of string/M theory vacua.
- We can test whether our hypothesized ensemble is accurate, by comparing with actual string/M theory constructions.
- We can find out what fraction of vacua out of our ensemble realize a given mechanism, or meet a specified phenomenological test.
- If we can decide how many vacua there are in an ensemble, and apply the known tests, we can estimate the number of Standard Model candidates in that ensemble.

We gave various results along these lines:

- We discussed a simple ensemble of quiver gauge theories which models the actual ensemble of Dirichlet brane world-volume theories coming from type II compactification on Calabi-Yau.

- We used this to make quantitative the idea that the Standard Model spectrum is "generic."

- We discussed a simplification of the ensemble of type II compactifications with flux, in which the total number of vacua can be computed. Our techniques can be pushed much farther, say to count non-supersymmetric vacua, or expected distances in configuration space between vacua.

We can compare to (Dine, 0210255), which suggests looking for "generic" properties of string vacua. Our proposal provides a quantitative definition of the word "generic."

Other accessible questions might include whether discrete symmetries are generic in string theory, whether hierarchies are generic, and so on.

While this approach does not directly lead to models, it has other advantages:

- The set of **all** vacua of string/M theory is likely to reveal many patterns and simplicities, not evident after restricting attention to a single gauge group or other feature.
- One can reason about ensembles using duality arguments. Two string constructions, which claim to provide dual realizations of the same class of models, should lead to the same ensembles. This provides a language in which to study $\mathcal{N} = 1$ (and nonsupersymmetric) duality.
- Models which "work," *i.e.* pass many phenomenological tests, are rare, and hard to find. In this approach, one can get evidence that models exist, without having to explicitly find them.
- Features/mechanisms which apply to $N >> 1$ vacua in an ensemble, can be considered **natural** in that ensemble. Thus, we can develop an idea of "stringy naturalness."

Such results can guide the search for models. More generally, we can try to find out whether or not string theory is predictive, and in what ways.