Dualities and Dimensional Reduction in Topological Quantum Order and Processing of Quantum Information

Emilio Cobanera

Department of Physics
Indiana University, Bloomington, IN

Rutgers, December 8, 2011
A vague philosophy,

“Interactions are more important than elementary degrees of freedom,”

and its technical implementation: **BOND ALGEBRAS**.

**Exact solvability** (Lie bond algebras)


**Dualities**

- Perturbation theory for strongly coupled systems
- Symmetries, transition points and boundaries of phase diagrams

1. Unified, generalized theory of quantum and classical dualities
2. Systematic derivation of topological degrees of freedom
3. Fermionization as a duality: derivation of the JW mapping
4. Gauge theories and TQO
5. Numerical applications: simplified STL for quantum Monte Carlo, dual boundary conditions,...

**Exact and Effective Dimensional reduction (holographic correspondences)**

1. Exact dimensional reduction as a duality
2. Tensor networks (DMRG) for two and three dimensional systems
3. When is a system "quasi" lower dimensional? Dim-red inequalities symmetry principles for dimensional reduction

Cobanera et. al. arXiv:1110.2179v1
[cond-mat.stat-mech]
**Non-Abelian dualities**

1. The character of a duality is not determined by the group of symmetries.
2. New dualities for the $S = 1/2$ Heisenberg model in any number of dimensions.
Model Building in Quantum Mechanics

**EDFs** ⇒ **basic interactions** \( \{ h_\Gamma \}_\Gamma \) ⇒

\[ H = \sum_\Gamma \lambda_\Gamma h_\Gamma \] ⇒ **Emergent EDFs**

The **BONDS** \( h_\Gamma \) are the “atomic constituents” of the Hamiltonian.

**Example:**

\[ \sigma_i^x, \sigma_i^z \Rightarrow \{ \sigma_i^x, \sigma_i^z \sigma_{i+1}^z \}_i \Rightarrow H_i = \sum_i [h_\sigma_i^x + J \sigma_i^z \sigma_{i+1}^z] \Rightarrow \text{Kinks} \]

**Bonds are SPARSE:** 

\[ [h_\Gamma, h_{\Gamma'}] = 0 \quad \text{for most } \Gamma' \]

Typically a consequence of **LOCALITY**
Our Philosophy: Interactions are more important than elementary degrees of freedom.

What are the EDFs? ↔ What is the algebra of the EDFs?
1. fermionic or bosonic algebra?
2. SU(N) spins, “Hopf spins”? etc. etc. etc. ...

What are the interactions? ↔ What is the algebra of interactions?

Definition

The bond algebra of \( H = \sum_{\Gamma} \lambda_{\Gamma} h_{\Gamma} \) is the von Neumann algebra of operators \( \mathcal{A}_{H} \) generated by the set of bonds \( \{ h_{\Gamma} \}_{\Gamma} \).

(Cobanera et. al., PRL 104, 020402 (2010))

\[
\mathcal{A}_{H} = \text{Linear Span} \{ 1, h_{\Gamma}, h_{\Gamma}^\dagger, h_{\Gamma} h_{\Gamma'}, h_{\Gamma'}^\dagger h_{\Gamma'}, h_{\Gamma}^\dagger h_{\Gamma}, h_{\Gamma} h_{\Gamma}, h_{\Gamma'}^\dagger h_{\Gamma'}, h_{\Gamma'} h_{\Gamma}, h_{\Gamma''}, \cdots \}
\]
**Idea:** Use bond algebras to compare Hamiltonians

\[ \Phi : A_{H_1} \rightarrow A_{H_2} \quad \text{one-to-one and onto} \]

\[
\begin{align*}
\Phi(1) & = 1, & \Phi(O^\dagger) & = \Phi(O)^\dagger, \\
\Phi(O_1 O_2) & = \Phi(O_1) \Phi(O_2), & \Phi(O_1 + \lambda O_2) & = \Phi(O_1) + \lambda \Phi(O_2).
\end{align*}
\]

**Definition**

\[ \Phi \text{ is a duality, and } H_1 \text{ is dual to } H_2 \text{ if } \Phi(H_1) = H_2 \]
Idea: Use bond algebras to compare Hamiltonians

\[ \Phi : A_{H_1} \rightarrow A_{H_2} \]  

one-to-one and onto

\[
\begin{align*}
\Phi(1) &= 1, \\
\Phi(O_1 O_2) &= \Phi(O_1) \Phi(O_2), \\
\Phi(O_1 + \lambda O_2) &= \Phi(O_1) + \lambda \Phi(O_2).
\end{align*}
\]

Definition

\( \Phi \) is a duality, and \( H_1 \) is dual to \( H_2 \), if \( \Phi(H_1) = H_2 \)

Theorem

\( \Phi(O) = UOU^\dagger \)  

Dualities are unitary equivalences!!!

Either  

\[ \begin{align*}
UU^\dagger &= U^\dagger U = 1, \\
UU^\dagger &= 1 \quad \text{and} \quad U^\dagger U = P = P^2.
\end{align*} \]
Transmutation of statistics I

\[ H_F = \sum_{i=1}^{N-1} \lambda (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) \]

Bonds: \( \{c_{i+1}^\dagger c_i \mid i = 1, \ldots, N - 1\} \)

\[ H_{XY} = \sum_{i=1}^{N-1} \lambda (\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-) \]

Bonds: \( \{\sigma_{i+1}^+ \sigma_i^- \mid i = 1, \ldots, N - 1\} \)

Very different EDFs, but isomorphic bond algebras:

\[ c_{i+1}^\dagger c_i \xrightarrow{\Phi_d} \sigma_{i+1}^+ \sigma_i^- \]

\( H_F \) is dual (unitarily equivalent!) to \( H_{XY} \)
Transmutation of statistics I

\[ H_F = \sum_{i=1}^{N-1} \lambda (c_i^+ c_{i+1} + c_{i+1}^+ c_i) \]

Bonds: \( \{c_{i+1}^+ c_i | i = 1, \ldots, N - 1\} \)

\[ H_{XY} = \sum_{i=1}^{N-1} \lambda (\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-) \]

Bonds: \( \{\sigma_{i+1}^+ \sigma_i^- | i = 1, \ldots, N - 1\} \)

Very different EDFs, but isomorphic bond algebras:

\[ c_{i+1}^+ c_i \xrightarrow{\Phi_d} \sigma_{i+1}^+ \sigma_i^- \]

\( H_F \) is dual (unitarily equivalent!) to \( H_{XY} \)

Dual Fermions:

\[ c_i \xrightarrow{\Phi_d} ??? \]
Transmutation of statistics II: Fermions as dual topological collective modes

1. **Enlarge** $A_F$ by adding $c_1$ to the set of bonds

   \[ c_2 = [c_1, c_1^\dagger c_2], \quad c_3 = [c_2, c_2^\dagger c_3], \quad \cdots, \quad c_N = [c_{N-1}, c_{N-1}^\dagger c_N] \]

2. **Extend** $\Phi_d$ so that all algebraic relations are preserved

   \[ c_1 \xrightarrow{\Phi_d} \sigma_1^- . \quad \text{Then, for } i = 2, \cdots, N \]

   \[ \Phi_d(c_2) = [\Phi_d(c_1), \Phi_d(c_1^\dagger c_2)] = [\sigma_1^-, \sigma_1^+ \sigma_2^-] = -\sigma_1^z \sigma_2^-, \quad \text{and so on...} \]

3. \[
\begin{align*}
  &c_i \xrightarrow{\Phi_d} \prod_{j=1}^{i-1} (-\sigma_j^z) \sigma_i^- \\
  &\equiv \widehat{c}_i \\
\text{JW transformation = dual fermions}
\end{align*}
\]
Fermionization can be understood as a duality in any number of dimensions, and
the corresponding JW transformation can be derived as a fermionic topological excitation.
Bond algebras can be used to
1. show that fermionization is not possible under certain conditions
2. look for dual representations of a model that are better suited for fermionization. Example: Two-dimensional Ising model in a transverse field

(Cobanera et. al., Adv. Phys 60, 679 (2011))
Are we really talking of dualities here? The quantum Ising chain

\[ H_I[h, J] = \sum_i \left[ h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z \right] \]

An infinite quantum Ising Chain

<table>
<thead>
<tr>
<th>Bond</th>
<th>anticommutes with</th>
<th>Bond²</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_i^x )</td>
<td>( \sigma_{i-1}^z\sigma_i^z )</td>
<td>( \sigma_i^z\sigma_{i+1}^z )</td>
</tr>
<tr>
<td>( \sigma_i^z\sigma_{i+1}^z )</td>
<td>( \sigma_i^x )</td>
<td>( \sigma_{i+1}^x )</td>
</tr>
</tbody>
</table>

\[ \sigma_i^x \xrightarrow{\Phi_d} \sigma_{i+1}^x \sigma_{i+1}^z \]
\[ \sigma_i^z\sigma_{i+1}^z \xrightarrow{\Phi_d} \sigma_{i+1}^x \]

\( H_I[h, J] \) is dual (unitarily equivalent!) to \( H_I[J, h] \)

\[ \Rightarrow E(J, h) = E(h, J) \Rightarrow J = h \] transition line
A duality is a mapping of bonds that preserves the algebra of interactions

\[ \mu_i^x \equiv \Phi_d(\sigma_i^x) = \sigma_i^z \sigma_{i+1} \]

\[ \mu_i^z \equiv \Phi_d(\sigma_i^z) = \Phi_d(\sigma_i^z \sigma_{i+1} \times \sigma_{i+1}^z \sigma_{i+2}^z \times \cdots) = \sigma_{i+1}^x \sigma_{i+1}^x \sigma_{i+2}^x \cdots \]

Dualities and TQO: The one-dimensional extended toric code

\[(i, 1) = \text{link connecting site } i \text{ and } i + 1\]

\[H_{\text{ETC}} = \sum_i \left[ h_z \sigma^z_{(i,1)} + h_x \sigma^x_{(i,1)} + J_x \sigma^x_{(i,1)} \sigma^x_{(i+1,1)} \right] \equiv A_{i+1}\]

\[
\begin{array}{c}
\bullet \\
\hline \\
\bullet \\
\hline \\
\bullet \\
\hline \\
\bullet \\
\hline \\
\bullet \\
\hline \\
\bullet
\end{array}
\]

(Tupitsyn et. al., Phy. Rev. B 82, 8 (2012); two dimensions)
Duality Mapping:

\[
\sigma_i^x \xrightarrow{\Phi_d} \sigma_{(i-1,1)}^x \sigma_{(i,1)}^x \equiv A_i, \quad \sigma_i^z \sigma_{(i,1)}^z \sigma_{i+1}^z \xrightarrow{\Phi_d} \sigma_{(i,1)}^z \quad \sigma_{(i,1)}^x \xrightarrow{\Phi_d} \sigma_{(i,1)}^x
\]

Have we lost degrees of freedom???
Dualities and Gauge Symmetries

\[ H_{ETC}^D = \sum_i \left[ J_x \sigma_i^x + h_z \sigma_i^z \sigma_{(i,1)}^z + h_x \sigma_i^x \sigma_{(i,1)}^x \right] \quad \mathbb{Z}_2 \text{ Higgs model} \]

(Fradkin and Shenker, Phys. Rev. D 19, 3682 (1979))

**Gauge Symmetries:**

\[ \sigma_i^x A_i = \sigma_{(i-1,1)}^x \sigma_i^x \sigma_{(i,1)}^x \]

A state \( \rho \) is physical if and only if \([\rho, \sigma_i^x A_i] = 0\)

**NOTICE:**

\[ \sigma_i^x A_i \xrightarrow{\Phi_d} A_i A_i = 1 \]

The duality changes the number of EDFs because it eliminates all the gauge symmetries.

\[ \Phi_d(\mathcal{O}) = U_d \mathcal{O} U_d^\dagger \quad U_d U_d^\dagger = 1 \quad U_d^\dagger U_d = P_{GI} \]
Both the $\mathbb{Z}_2$ Higgs model and (extended) toric code model have natural (canonical) generalizations to any number of dimensions and arbitrary Abelian group $G$. If the group is continuous we may be able to take the continuum limit.

They are always dual, and the phase diagrams of some of these generalizations are under investigation. In two dimensions, the continuum limit of the ETC model with group $\mathbb{R}$ is the Stückelberg model of mass generation.

In two dimensions, the duality still holds on more general lattices like the honeycomb lattice. It suggests some interesting questions on the stability of some string-net topological phases.

The Big Challenge: What if $G$ is non-Abelian?
Strong Coupling/Weak Coupling

STL decomposition/Feynmann’s path integral

\[ Z_E = \sum_{\{\phi_1\}, \ldots, \{\phi_N\}} \langle \phi_1 | e^{-1/N} H | \phi_2 \rangle \langle \phi_2 | e^{-1/N} H | \phi_3 \rangle \cdots \langle \phi_N | e^{-1/N} H | \phi_1 \rangle = \text{Tr} \left( e^{-1/N} H \right)^N \]

\[ Z_E = \text{Tr} \left( e^{-1/N} H \right)^N = \text{Tr} \left( e^{-1/N} H^D \right)^N = Z_E^D \]

Bond-Algebraic Classical Dualities

\[ Z = \text{Tr} \left( T_1 \cdots T_s \right)^N \quad T_i = \prod_{\Gamma} t_{i\Gamma} \]

Bond algebra \( \mathcal{A}_Z \) : algebra generated by the \( \{ t_{i\Gamma} \} \)
Strong Coupling/Weak Coupling dualities are “classical” descendants of quantum dualities

\[ Z_1(K, \tilde{h}) = \sum_{\{\sigma_i\}} \exp \left[ \sum_{i=1}^{N} (K \sigma_i \sigma_{i+1} + \tilde{h} \sigma_i) \right] = \text{Tr} (T_1 T_2)^N \]

\[ T_1 = e^K + e^{-K} \sigma^x, \quad T_2 = e^{\tilde{h}} \sigma^z = \cosh(\tilde{h}) + \sinh(\tilde{h}) \sigma^z, \]

\[ T_1^D = e^K + e^{-K} \sigma^z = A \ e^{\tilde{h}^*} \sigma^z, \quad T_2^D = e^{\tilde{h}} \sigma^x = B (e^{K^*} + e^{-K^*} \sigma^x), \]

\[ \sinh(2K) \sinh(2\tilde{h}^*) = 1, \quad \sinh(2K^*) \sinh(2\tilde{h}) = 1 \]
Bond-algebraic dualities are unitary transformations

\[
\frac{Z_1(K, \tilde{\hbar})}{(2 \sinh(2\tilde{\hbar}))^{N/2}} = \frac{\text{Tr} (T_1 T_2)^N}{(2 \sinh(2\tilde{\hbar}))^{N/2}} = \frac{\text{Tr} (T_2^D T_1^D)^N}{(2 \sinh(2\tilde{\hbar}))^{N/2}} = \frac{Z_1(K^*, \tilde{\hbar}^*)}{(2 \sinh(2\tilde{\hbar}^*))^{N/2}}
\]

Classical self-dual line defined by \(\tilde{\hbar}^* = \tilde{\hbar}\) and \(K^* = K\)

\[
\sinh(2K) \sinh(2\tilde{\hbar}) = 1
\]

Can only be critical if \(\tilde{\hbar} = 0\) \(\Rightarrow K \to \infty\), i.e., at zero temperature.
Quantum dualities are “mapped” to strong coupling/weak coupling dualities of partition functions/Euclidean path integrals.

Many possible dualities! Only one corresponds to the standard classical duality based on the Fourier transform.
Holographies and dualities

- Dimensional reduction and holographic correspondences qualify those situations when the “apparent”, geometric dimension of a system is not the dimension that best characterizes its response to probes and information-theoretic aspects.
  
  1. **Restricted dynamics from conservation laws** (sliding dynamics)
  2. **Restricted dynamics from special couplings and interactions** (layered systems)
  3. **Kaluza-Klein compactification** (string theory)
  4. **Gauge-gravity dualities** (AdS-CFT correspondence)

- Bond algebras display an *internal connectivity* that may or may not reflect the apparent geometric connectivity of the model.

- **Bond-algebraic dualities can change the dimension of a system.**
An fcc lattice has exactly one octahedron per lattice site. Define the "octahedron operator"

$$O_m = \sigma^x_{m+a_1-a_2} \sigma^x_{m+a_3} \sigma^y_{m+e_2} \sigma^y_{m+a_3-a_2} \sigma^z_{m+a_1}$$

$$H_{xyz} = -J \sum_m O_m$$

displays topological quantum order. Its bond algebra is commutative.
The fcc lattice is quadripartite. If it satisfies periodic boundary conditions, then \((i = 1, 2)\)

\[ \prod_{m \in A_i} O_m = \prod_{m \in B_i} O_m = 1. \]

These constraints further structure the commutative bond algebra of the model.

The XYZ model is dual to **four decoupled, periodic, Ising chains**.

(Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech])
Many models of TQO are dual to one dimensional models. This is not because their bond algebra is commutative, but rather because the constraints are simple. We can add non-commutativity and preserve dimensional reduction.

**Thermal fragility:** a periodic Ising chains display short autocorrelation times at any finite temperature, *regardless of its size*. Models of TQO that display this type of dimensional reduction may not be good quantum memories. Most famously,

1. The Toric Code, Honeycomb toric code, topological color codes, and
2. the XYZ model just discussed

But, exact dimensional reduction is a rare. How can we quantify and exploit **approximate or effective** dimensional reduction?

Cobanera et. al., arXiv:1110.2179v1 [cond-mat.stat-mech]
Effective Dimensional Reduction in Classical Systems

(Batista and Nussinov, Phys. Rev. B 72, 045137 (2005))

\[ \phi(x) = \begin{cases} 
\phi_0(x) & \text{if } x \in \Gamma \\
\psi(x) & \text{if } x \in \bar{\Lambda} 
\end{cases} \]

\[ f[\phi] = f[\phi_0] \quad \text{localized observable} \]

\[ \langle f \rangle^D = \sum_{\{\psi\}} \sum_{\{\phi_0\}} f[\phi_0] \frac{e^{-\beta E[\phi_0,\psi]}}{Z} = \sum_{\{\psi\}} \frac{z[\psi]}{Z} \sum_{\{\phi_0\}} f(\phi_0) e^{-\beta E[\phi_0,\psi]} z[\psi] \]

\[ \langle f \rangle^d_l = \min_{\psi} \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\min}], \quad \langle f \rangle^d_u = \max_{\psi} \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\max}] \]

\[ \langle f \rangle^d_l \leq \langle f \rangle^D \leq \langle f \rangle^d_u \]

\[ \langle f \rangle^d_l : E_l[\phi_0, \psi_{\min}] \quad \text{and} \quad \langle f \rangle^d_u : E_u[\phi_0, \psi_{\max}] \quad \text{LOCAL effective theories} \]
Consider a system on a volume $\Lambda$ with distinguishable bulk $\bar{\Lambda}$ and boundary $\Gamma$:

$$\mathcal{H}_\Lambda = \mathcal{H}_\Gamma \otimes \mathcal{H}_{\bar{\Lambda}}$$

We can write an arbitrary state as

$$\rho = \sum_i \lambda_i \rho_{\Gamma i} \otimes \rho_{\bar{\Lambda} i}, \quad \lambda_i \in \mathbb{R}, \quad \sum_i \lambda_i = 1.$$ 

If the $\lambda_i$ are all positive, the state is separable (unentangled).
**Arbitrary state** 
\[ \rho = \sum_i \lambda_i \rho_{\Gamma i} \otimes \rho_{\bar{\Lambda} i}, \quad \lambda_i \in \mathbb{R}, \quad \sum_i \lambda_i = 1. \]

\[ f = f_\Gamma \otimes 1_\bar{\Lambda} \] localized on the boundary

**Theorem**

\[ L_+ \langle f \rangle^+_l - L_- \langle f \rangle^-_l \leq \text{Tr} \wedge (\rho f) \leq L_+ \langle f \rangle^+_u - L_- \langle f \rangle^-_u. \]

Where \( L_+ = \sum_{i_+} \lambda_{i_+}, \quad L_- = \sum_{i_-} |\lambda_{i_-}| \) are both positive,

\[ \langle f \rangle^+_u \equiv \max_{i_+} \text{Tr} \Gamma (\rho_{\Gamma i_+} f_\Gamma), \quad \langle f \rangle^-_u \equiv \min_{i_-} \text{Tr} \Gamma (\rho_{\Gamma i_-} f_\Gamma), \]

\[ \langle f \rangle^+_l \equiv \min_{i_+} \text{Tr} \Gamma (\rho_{\Gamma i_+} f_\Gamma), \quad \langle f \rangle^-_l \equiv \max_{i_-} \text{Tr} \Gamma (\rho_{\Gamma i_-} f_\Gamma). \]

If state \( \rho \) is unentangled, then \( L_- = 0 \) and \( L_+ = 1 \):

\[ \langle f \rangle^+_l \leq \text{Tr} \wedge (\rho f) \leq \langle f \rangle^+_u \]

Cobanera et. al., arXiv:1110.2179v1 [cond-mat.stat-mech]
Entanglement-based inequalities are ideal to establish a connection to classical notions of effective dimensional reduction.

There are other inequalities that are better suited to purely quantum-mechanical investigations (Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech]).

Effective dimensional reduction combined with low dimensional gauge like symmetries and results like Elitzur’s or Mermin-Wagner-Coleman theorem can put strong constraints on symmetry breakdown in higher dimensions.

Effective dimensional reduction may help to assess the viability of realistic proposals for topological quantum memories.
Summary and conclusions

Bond algebras are useful!!!

Bond-Algebraic dualities are one of the best developed applications of bond algebras. They work well with TQO because they can handle gauge symmetries easily.

Bond algebras encode the “true” dimensionality of a system as witnessed by its interactions, and a duality can then unveil exact dimensional reduction.

Exact dimensional reduction is rare, so we propose a set of inequalities to quantify effective dimensional reduction. They may be of consequence to quantum information processing.
Acknowledgments, and Thank you!

**Gerardo Ortiz**, Indiana University, Bloomington IN

**Zohar Nussinov**, Washington University, St Louis MO

**Emanuel Knill**, NIST, Boulder CO
Thank you!
\[ Z_1[K] = \sum_{\sigma_r} \exp \left[ K \sum_{r} \sum_{\nu=1,2} \sigma_{r+e_{\nu}} \sigma_r \right] \]

\[ K = -\beta J = -J/k_B T \geq 0, \text{ ferromagnetic} \]

One discrete global symmetry

\[ \sigma_r \mapsto -\sigma_r \]

that can be broken. What is the critical temperature?
Bond algebras for classical dualities

If we introduce the row-to-row transfer matrices

\[ T_0 = \prod_i \exp[K \sigma^z_i \sigma^z_{i+1}], \quad T_1 = \prod_i (e^K + e^{-K} \sigma^x_i) \]

then we can write

\[ \mathcal{Z}_I[K] = \text{Tr} \left[ T_1 T_0 T_1 T_0 \cdots T_1 T_0 \right] = \text{Tr} \left[ (T_1 T_0)^N \right] \]

provided we agree to compute the trace in the diagonal basis for the \( \sigma^z_i \). \( N \) determines the height (number of rows) of the system. The bond algebra is the same as before!

\[ |\sigma_{i-1,j+1}>, |\sigma_{i,j+1}>, |\sigma_{i+1,j+1}>, \]

row \( j+1 \)

\[ T_{1,i-1}, T_{1,i}, T_{1,i+1} \]

row \( j \)

\[ |\sigma_{i-1,j}>, |\sigma_{i,j}>, |\sigma_{i+1,j}>, \]

\[ = \prod_i \exp[-\beta J \sigma_i,j \sigma_{i,j+1}] \]
The self-duality of Kramers and Wannier and the critical temperature of the Ising model

\[ T_0 \xrightarrow{\Phi_d} T_0^D = \prod_i \exp[K\sigma_i^x], \quad T_1 \xrightarrow{\Phi_d} T_1^D = \prod_i (e^K + e^{-K}\sigma_i^z\sigma_{i+1}^z) \]

This is a UNITARY TRANSFORMATION. Hence

\[ \mathcal{Z}_I[K] = \text{Tr} \left[ (T_1 T_0)^N \right] = \text{Tr} \left[ (T_1^D T_0^D) \right] \equiv \mathcal{Z}_I^D \]
The self-duality of Kramers and Wannier and the critical temperature of the Ising model

\[ T_0 \xrightarrow{\Phi_d} T_0^D = \prod_i \exp[K\sigma_i^x], \quad T_1 \xrightarrow{\Phi_d} T_1^D = \prod_i (e^K + e^{-K}\sigma_i^z\sigma_{i+1}^z) \]

This is a **UNITARY TRANSFORMATION**. Hence

\[ \mathcal{Z}_I[K] = \text{Tr} \left[ (T_1 T_0)^N \right] = \text{Tr} \left[ (T_1^D T_0^D) \right] \equiv \mathcal{Z}_I^D \]

Next, a little bit of math shows that

\[ \mathcal{Z}_I[K] = \mathcal{Z}_I^D \propto \mathcal{Z}_I[K^*], \quad K^* = -\frac{1}{2} \ln \tanh(K) \]

A weak coupling-strong coupling transformation has emerged!
The self-duality of Kramers and Wannier and the critical temperature of the Ising model

\[ T_0 \xrightarrow{\Phi_d} T_0^D = \prod_i \exp[K \sigma_i^x], \quad T_1 \xrightarrow{\Phi_d} T_1^D = \prod_i (e^K + e^{-K} \sigma_i^z \sigma_{i+1}^z) \]

This is a \textbf{UNITARY TRANSFORMATION}. Hence

\[ \mathcal{Z}_I[K] = \text{Tr} \left[ (T_1 T_0)^N \right] = \text{Tr} \left[ (T_1^D T_0^D) \right] \equiv \mathcal{Z}_I^D \]

Next, a little bit of math shows that

\[ \mathcal{Z}_I[K] = \mathcal{Z}_I^D \propto \mathcal{Z}_I[K^*], \quad K^* = -\frac{1}{2} \ln \tanh(K) \]

A weak coupling-strong coupling transformation has emerged! If there is only one critical point, then its value must be

\[ K_c = \frac{1}{2} \ln(1 + \sqrt{2}) \]
Duality are not symmetries, but

\[ U_d H_1 U_d^\dagger = H_2 \]
\[ \tilde{U} d H_1 \tilde{U} d^\dagger = H_2 \]

\[ (U_d \tilde{U} d^\dagger) H_1 (U_d \tilde{U} d^\dagger)^\dagger = H_1 \]
\[ (U_d \tilde{U} d^\dagger) H_2 (U_d \tilde{U} d^\dagger)^\dagger = H_2 \]

Self-dualities

\[ U_d H_1[\lambda_1, \lambda_2, \ldots] U_d^\dagger = H_2[\lambda^*_1, \lambda^*_2, \ldots] \]

become extra, discrete, non-trivial symmetries at self-dual points where \( \lambda_i = \lambda^*_i \).
Symmetries and dualities

1. **Dualities are not symmetries**, but
2. they are not unique and thus may **reveal** hidden symmetries. If

\[ U_d H_1 U_d^\dagger = H_2 \quad \text{and} \quad \tilde{U}_d H_1 \tilde{U}_d^\dagger = H_2 \]

then

\[ (U_d^\dagger \tilde{U}_d) H_1 (U_d^\dagger \tilde{U}_d)^\dagger = H_1 \quad \text{and} \quad (U_d \tilde{U}_d^\dagger) H_2 (U_d \tilde{U}_d^\dagger)^\dagger = H_2 \]
1. **Dualities are not symmetries**, but they are not unique and thus may **reveal** hidden symmetries. If

\[ U_d H_1 U_d^\dagger = H_2 \quad \text{and} \quad \tilde{U}_d H_1 \tilde{U}_d^\dagger = H_2 \]

then

\[ (U_d^\dagger \tilde{U}_d) H_1 (U_d^\dagger \tilde{U}_d)^\dagger = H_1 \quad \text{and} \quad (U_d \tilde{U}_d^\dagger) H_2 (U_d \tilde{U}_d^\dagger)^\dagger = H_2 \]

2. **Self-dualities**

\[ U_d H[\lambda_1, \lambda_2, \cdots] U_d^\dagger = H[\lambda_1^*, \lambda_2^*, \cdots] \]

become **extra, discrete, non-trivial** symmetries at self-dual points where \( \lambda_i = \lambda_i^* \).
The self-duality of the Ising model is the square-root of a translation by one unit to the right, $\mathcal{U}_d^2 = T(1)$:
The self-duality of the Ising model is the square-root of a translation by one unit to the right, $U_d^2 = T(1)$:

It becomes an **extra symmetry** of the model’s **self-dual point**

$$U_d H[h, J = h] U_d^\dagger = H[h, J = h]$$

where the phase transition occurs.
Confinement and topological quantum order: The new face of an old phase diagram

The $\mathbb{Z}_2$ Higgs model $(B_{(r,3)} \equiv \sigma_z^{r,1}\sigma_z^{r+e_1,2}\sigma_z^{r+e_2,1}\sigma_z^{r,2})$:

$$H_{AH} = \sum_r \left( J_x \sigma^r_x + J_z B_{(r,3)} \right) + \sum_r \sum_{\nu=1,2} \left( h_z \sigma^r_z \sigma_z^{r,\nu} \sigma_z^{r+e_\nu} + h_x \sigma^r_x \sigma^{(r,\nu)} \right)$$
Symmetries and phase diagram of the $\mathbb{Z}_2$ Higgs model

The gauge symmetries are $G_r \equiv \sigma_r^x A_r$, with

$$A_r \equiv \sigma_{(r,1)}^x \sigma_{(r,2)}^x \sigma_{(r-e_1,1)}^x \sigma_{(r-e_2,2)}^x$$

the star operator.

There can be no spontaneous breakdown of gauge symmetries (Elitzur’s theorem). But we can try to get rid of them to have easier access to the model’s phase diagram. Dualities!
The bond algebra of the $\mathbb{Z}_2$ Higgs model has at least one dual representation that “leaps to the eye:”

$$\sigma_r^x \xrightarrow{\Phi_d} A_r$$

$$\sigma_r^z \sigma_{(r,\nu)}^z \sigma_r^z \xrightarrow{\Phi_d} \sigma_{(r,\nu)}^z$$

$$B_{(r,3)} \xrightarrow{\Phi_d} B_{(r,3)}$$

$$\sigma_{(r,\nu)}^x \xrightarrow{\Phi_d} \sigma_{(r,\nu)}^x$$

The Dual Hamiltonian

$$H_{AH} \xrightarrow{\Phi_d} H_{ETC} = \sum_r \left[ (J_x A_r + J_z B_{(r,3)}) + \sum_{\nu=1,2} (h_z \sigma_{(r,\nu)}^z + h_x \sigma_{(r,\nu)}^x) \right]$$

The Higgs model is dual to the Toric Code model in a magnetic field. But this extended toric code has no gauge symmetries!!!! Where did they go?

$$G_r = \sigma_r^x A_r \xrightarrow{\Phi_d} A_r A_r = 1$$

The duality has solved completely the gauge constraints!!!!
What is the link between TQO and dimensional reduction?

Models of TQO typically display \(d\)-dimensional gauge-like symmetries, that combined with dimensional-reduction techniques can yield important information about phase diagrams.

\[
H_{\text{POC}} = -\sum_{r}(J_1 \sigma^x_r \sigma^x_{r+e_1} + J_2 \sigma^y_r \sigma^y_{r+e_2})
\]

\[
X_{i_1} = \prod_{i_2} \sigma^x_{i_1,i_2}, \quad Y_{i_2} = \prod_{i_1} \sigma^y_{i_1,i_2}
\]

\(d\) gauge-like symmetries have been proposed to be the symmetry principle underlying both TQO and dimensional reduction. (Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech])
Exact dualities for finite systems require special boundary conditions, called dual boundary conditions.
Exact dualities for finite systems require special boundary conditions, called dual boundary conditions. Dual boundary conditions are model-specific, and can be computed on a case-by-case basis straight from the bond algebra of the finite systems under consideration.

$$H_l^N = \hbar \sigma_l^x + \sum_{i=2}^{N} [J \sigma_{i-1}^z \sigma_i^z + \hbar \sigma_i^x]$$

$$H_l^N$$  NOT Self-Dual, $E(h, J) \neq E(J, h)$

$$H_l^N + J \sigma_1^z$$  Self-Dual, $E(h, J) = E(J, h)$