The S-Matrix in Twistor Space

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outline

- motivation: holography + twistors
- review: spinor helicity + BCFW recursion
- tree amplitudes in twistor space
- Hodges diagrams + inverse soft factors
- the holographic equation

motivation

A primary motivation for this work is holography.

After all, the S-matrix is a boundary observable encoding the overlap of $|T=-\infty\rangle$ with $|T=\infty\rangle$.

Broadly speaking, the "bulk" consists of off-shell states which make locality and LI manifest:

$$S = \int d^4x \, \mathcal{L}(x)$$

dual theory?

The price of locality and LI is redundancy:

- gauge + diffeomorphism invariance
- auxiliary fields in off-shell SUSY
- field reparameterization freedom

Is there an alternative to the QFT description?
Is there a theory dual to flat space?

on-shell methods

BCFW recursion relations construct the S-matrix from purely on-shell data:

1) Define M_3, the on-shell 3pt amplitude.

2) Recursively construct M_n from $M_{m<n}$.

Caveats: BCFW only valid at tree-level and for certain theories. And it is not a theory!

twistors $\longrightarrow$ dual theory

Witten: the perturbative expansion of $\mathcal{N} = 4$ SYM is computed by a dual topological string theory in twistor space.

Applying the same "data-driven" approach, we find that the natural home for the S-matrix is in fact ambi-twistor space.

"Data" in hand, we argue that **there is a new rule for building the S-matrix!**

spinor helicity formalism

The S-matrix of massless particles in 4d is naturally represented using the spinor helicity formalism.

The premise is to represent each null momentum vector by a bi-spinor:

$$p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$

where in $(2,2)$ signature, λ and $\tilde{\lambda}$ are real and independent spinors.

lorentz and little group

For particles $\{i\}$ labeled by $\{\lambda_i, \tilde{\lambda}_i\}$, the obvious LI quantities are angle and square brackets:

$$\begin{aligned} \lambda_{i\alpha}\epsilon^{\alpha\beta}\lambda_{j\beta} &= \langle ij \rangle = -\langle ji \rangle \\ \tilde{\lambda}_i{}^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_j{}^{\dot{\beta}} &= [ij] = -[ji] \end{aligned}$$

These are covariant under action of the little group:

$$\lambda_i \to t_i \lambda_i, \qquad \tilde{\lambda}_i \to t_i^{-1} \tilde{\lambda}_i$$

which by definition leaves $p_i = \lambda_i \tilde{\lambda}_i$ invariant.

on-shell amplitudes

For particles $\{i\}$ of spin s and helicity $\{h_i\}$, the on-shell amplitude takes the form

$$M(\{\lambda_i, \tilde{\lambda}_i; h_i\}) = \mathcal{M}(\{\lambda_i, \tilde{\lambda}_i; h_i\})\delta^4\left(\sum_i \lambda_i \tilde{\lambda}_i\right)$$

where under the little group

$$M(\{t_i\lambda_i, t_i^{-1}\tilde{\lambda}_i; h_i\}) = t_i^{-2sh_i} M(\{\lambda_i, \tilde{\lambda}_i; h_i\})$$

some example amplitudes

YM tree amplitudes:

$$\mathcal{M}(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \qquad \mathcal{M}(1^+2^+3^-) = \frac{[12]^3}{[23][31]}$$

$$\mathcal{M}(1^+2^-3^+4^-) = \frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$\mathcal{M}(1^+2^-3^+4^-5^+6^-) =$$

$$\frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle\langle 6|p_1+p_2|3]\langle 4|p_2+p_3|1](p_1+p_2+p_3)^2}$$

$$+\{i\to i+2\}+\{i\to i+4\}$$

some more example amplitudes

There are closed formulae for all MHV (maximally helicity violating) and anti-MHV amplitudes:

$$\mathcal{M}(1^+2^+\dots i^-\dots j^-\dots n^+) = \frac{\langle ij\rangle^4}{\prod_{k=1}^{n}\langle k\ k+1\rangle}$$

$$\mathcal{M}(1^-2^-\dots i^+\dots j^+\dots n^-) = \frac{[ij]^4}{\prod_{k=1}^{n}[k\ k+1]}$$

spinor helicity $\rightarrow$ no polarizations!

BCFW

BCFW constructs the S-matrix recursively. In $(3,1)$, we shift i and j by a complex parameter z:

$$\lambda_i(z) = \lambda_i + z\lambda_j, \quad \tilde{\lambda}_j(z) = \tilde{\lambda}_j - z\tilde{\lambda}_i$$

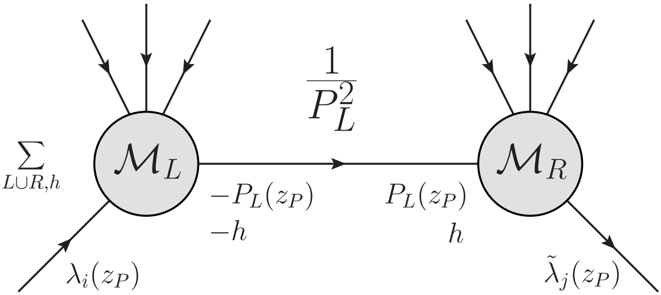
$\mathcal{M}$(z) is complexified. BCFW = Cauchy's theorem:

$$\mathcal{M}(0) = \oint \frac{dz}{z}\mathcal{M}(z) = \sum_{z_P} \frac{\mathcal{M}(z_P)}{z_P}$$

In YM and gravity, there is no pole at $z = \infty$ as long as $(h_i, h_j) \neq (-,+)$.

BCFW

Summing over z_P yields the BCFW reduction of $\mathcal{M}$:



where the pole is at $z_P = -\frac{P_L^2}{2[i|P_L|j\rangle}$.

With maximal SUSY, all h are smoothly labeled by η. For BCFW, shift $\eta_i(z) = \eta_i + z\eta_j$ and replace $\sum_h$ with $\int d^{\mathcal{N}}\eta$.

counting terms

Feynman diagrams are very redundant!

\# of terms in the *n*pt amplitude:

n legs	4	5	6	7	8
Feynman diagrams	4	25	220	2485	34300
BCFW recursion	1	1	3	6	20

Real world calculations are much faster and the final expressions are much more compact.

changing signatures

In $(2,2)$, we instead shift by a real parameter τ:

$$\lambda_i(\tau) = \lambda_i + \tau\lambda_j, \quad \tilde{\lambda}_j(\tau) = \tilde{\lambda}_j - \tau\tilde{\lambda}_i$$

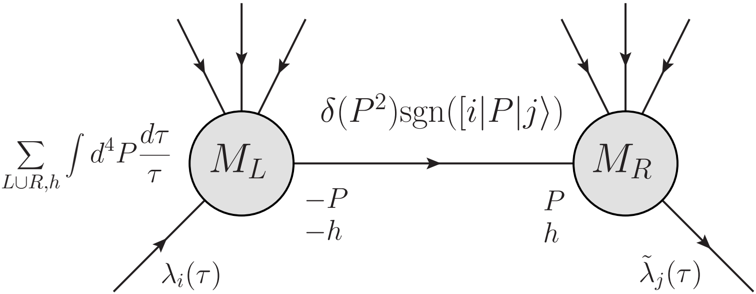
So $1/P_L^2$ can be expressed in a fully on-shell form:

$$\frac{M_L(\tau_P)M_R(\tau_P)}{P_L^2} = \int \frac{d\tau}{\tau}\delta(P^2)\mathrm{sgn}([i|P|j\rangle)M_L(\tau)M_R(\tau)$$

where $P = P_L + \tau\lambda_i\tilde{\lambda}_j$ on the support of M_L.

on-shell BCFW

In $(2,2)$, there is a fully on-shell form of BCFW:



Momentum conservation is built into $M_{L,R}$.

wave mechanics 101

Since the BCFW momentum shift is real

$$\lambda_i(\tau) = \lambda_i + \tau\lambda_j, \quad \tilde{\lambda}_j(\tau) = \tilde{\lambda}_j - \tau\tilde{\lambda}_i$$

we know what to do from wave mechanics:

$$\begin{aligned} f(x - vt) &= \int dk\, e^{ik(x-vt)}\tilde{f}(k) \\ \tilde{f}(t) &= e^{-ikvt}\tilde{f}(0) \end{aligned}$$

When you see a shifted variable, fourier transform it!

wave mechanics 101

A fourier transform from $\lambda_i \to \tilde{\mu}_i$ and $\tilde{\lambda}_j \to \mu_j$ is precisely a transform into ambi-twistor space!

$$M(\lambda_i(\tau), \tilde{\lambda}_j(\tau)) = \int d\tilde{\mu}_i d\mu_j e^{i\lambda_i(\tau)\tilde{\mu}_i} e^{i\tilde{\lambda}_j(\tau)\mu_j} \tilde{M}(\tilde{\mu}_i, \mu_j)$$

The shift becomes a phase:

$$\tilde{M}(\tau) = e^{i\tau(\lambda_j\tilde{\mu}_i - \tilde{\lambda}_i\mu_j)} \tilde{M}(0) = e^{i\tau W_i Z_j} \tilde{M}(0)$$

where $W_i Z_j$ is the natural LI in ambi-twistor space!

ambi-twistor space

Each particle is represented either by a twistor

$$\{\lambda, \tilde{\lambda}\} \to \{\lambda, \mu\} \equiv Z^A$$

or by a dual twistor

$$\{\lambda, \tilde{\lambda}\} \to \{\tilde{\mu}, \tilde{\lambda}\} \equiv W_A$$

which are both vectors of the $SL(4, \mathbb{R})$ conformal group. The natural invariant is

$$W_A Z^A \equiv WZ = \lambda\tilde{\mu} - \tilde{\lambda}\mu$$

ambi-twistor space

There is also the LI, conformal breaking quantity:

$$Z_i{}^A I_{AB} Z_j{}^B = \langle ij \rangle, \quad W_{iA} I^{AB} W_{jB} = [ij]$$

Under the little group Z and W transform as

$$Z \to tZ, \quad W \to t^{-1} W$$

while the amplitude transforms as

$$M(tZ; h) = t^{-2sh-2} M(Z; h)$$
$$M(t^{-1}W; h) = t^{-2sh+2} M(W; h)$$

ambi-dexterity

We can "ambi-dextrously" transform between the Z and W basis for any given particle:

$$M(W) = \int d^4Z \; e^{iWZ} M(Z)$$

BCFW suggests $\{- \leftrightarrow Z\}$ **and** $\{+ \leftrightarrow W\}$**!**

Before taking BCFW into twistor space, let's first see what some amplitudes look like in twistor space.

3pt YM

The anti-MHV YM 3pt amplitude is

$$\begin{aligned} M(1^+2^+3^-) &= \frac{[12]^3}{[23][31]}\delta^4\left(\sum_i \lambda_i\tilde{\lambda}_i\right) \\ &= \frac{[12]^3}{[23][31]}\int d^4X_{a\dot{a}}e^{iX(\sum_i \lambda_i\tilde{\lambda}_i)} \end{aligned}$$

To go to twistor space, simply fourier transform:

$$M(W_1^+, W_2^+, Z_3^-) = \int d^2\lambda_1 d^2\lambda_2 d^2\tilde{\lambda}_3 e^{i(\lambda_1\tilde{\mu}_1+\lambda_2\tilde{\mu}_2+\tilde{\lambda}_3\mu_3)}M(1^+2^+3^-)$$

3pt YM

Two of the integrals are trivial:

$$M(W_1^+, W_2^+, Z_3^-) \;=\; [12]^3 \int d^4X \int d^2\tilde{\lambda}_3 \frac{e^{i\tilde{\lambda}_3(\mu_3 + X\lambda_3)}}{[23][31]}$$
$$\delta^2(\tilde{\mu}_1 + X\tilde{\lambda}_1)\delta^2(\tilde{\mu}_2 + X\tilde{\lambda}_2)$$

For the $\tilde{\lambda}_3$ integral define $\tilde{\lambda}_3 = a\tilde{\lambda}_1 + b\tilde{\lambda}_2$. Finally:

$$M(W_1^+, W_2^+, Z_3^-) = \mathrm{sgn}([12]) \int \frac{da}{a}\frac{db}{b} e^{i(aW_1 Z_3 + bW_2 Z_3)}$$

How do we regulate the divergence?

principle value prescription

To determine the regularization, use the little group:

$$M(t_1^{-1}W_1^+, t_2^{-1}W_2^+, t_3 Z_3^-) = M(W_1^+, W_2^+, Z_3^-)$$

Thus, the only consistent prescription is PV:

$$\frac{1}{a} \to \frac{1}{2}\left(\frac{1}{a+i\epsilon} + \frac{1}{a-i\epsilon}\right)$$

which means that $\int da\, e^{iax}/a = i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(x)$.

3pt YM

Thus, the 3pt amplitude becomes

$$
\begin{aligned}
M(W_1^+, W_2^+, Z_3^-) &= \mathrm{sgn}(W_1 I W_2)\mathrm{sgn}(W_1 Z_3)\mathrm{sgn}(W_2 Z_3) \\
M(Z_1^-, Z_2^-, W_3^+) &= \mathrm{sgn}(Z_1 I Z_2)\mathrm{sgn}(Z_1 W_3)\mathrm{sgn}(Z_2 W_3)
\end{aligned}
$$

The 3pt amplitude in YM theory is 1 and -1.

All non-trivial dependence in momentum space arises from a Jacobian!

4pt YM, 3pt gravity

The YM 4pt amplitude is:

$$M(W_1^+ Z_2^- W_3^+ Z_4^-) = \mathrm{sgn}(W_1 Z_2)\mathrm{sgn}(W_1 Z_4)\mathrm{sgn}(W_3 Z_2)\mathrm{sgn}(W_3 Z_4)$$

The gravity 3pt amplitudes are:

$$\begin{aligned} M_{\mathrm{grav}}(W_1^+ W_2^+ Z_3^-) &= |W_1 I W_2||W_1 Z_3||W_2 Z_3| \\ M_{\mathrm{grav}}(Z_1^- Z_2^- W_3^+) &= |Z_1 I Z_2||Z_1 W_3||Z_2 W_3| \end{aligned}$$

the link representation

There is a convenient representation for amplitudes:

$$M(\{W_I, Z_J\}) = \int \left(\prod_{IJ} dc_{IJ}\right) \hat{M}(\{c_{IJ}; \lambda_J, \tilde{\lambda}_I\}) \, e^{ic_{IJ} W_I Z_J}$$

For example:

$$\begin{aligned}
\hat{M}(1^+2^+3^-) &= \frac{\mathrm{sgn}([12])}{c_{13} c_{23}} \\
\hat{M}(1^+2^-3^+4^-) &= \frac{1}{c_{12} c_{14} c_{32} c_{34}} \\
\hat{M}(1^+2^-3^+4^-5^+6^-) &= \frac{\mathrm{sgn}([13]\langle 46\rangle)}{c_{12} c_{32} c_{14} c_{54} c_{36} c_{56} (c_{14} c_{36} - c_{16} c_{34})} + \ldots
\end{aligned}$$

the link representation

Going back to momentum space yields

$$M = \int \left(\prod_{IJ} dc_{IJ} \right) \ \hat{M} \, \delta^2(\lambda_I - c_{IJ}\lambda_J)\delta^2(\tilde{\lambda}_J + c_{IJ}\tilde{\lambda}_I)$$

which is very reminiscent of the RSV formula!

The δ^4 has been "factored" into δ^2's.

BCFW = solving linear equations of the c_{IJ}'s.

$\mathcal{N} = 4$ SYM

$\mathcal{N} = 0$ tree amplitudes are obtained from $\mathcal{N} = 4$ tree amplitudes by fixing external legs to be gluons.

In maximal SUSY, every state is also labeled by an on-shell superspace variable: η_I or $\bar{\eta}^I$, $1 \leq I \leq \mathcal{N}$.

The super-twistor variables and LI invariants are:

$$\mathcal{Z} = \begin{pmatrix} Z^A \\ \eta_I \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} W_A \\ \tilde{\eta}^I \end{pmatrix}$$

$$\mathcal{W}\mathcal{Z} = WZ + \tilde{\eta}\eta, \quad \mathcal{W}_i I \mathcal{W}_j = [ij], \quad \mathcal{Z}_i I \mathcal{Z}_j = \langle ij \rangle$$

$\mathcal{N} = 4$ SYM

The MHV and anti-MHV 3pt amplitudes become:

$$\begin{aligned} M^{+}(\mathcal{W}_1, \mathcal{W}_2, \mathcal{Z}_3) &= \operatorname{sgn}(\mathcal{W}_1 I \mathcal{W}_2)\, \operatorname{sgn}(\mathcal{W}_1 \mathcal{Z}_3)\, \operatorname{sgn}(\mathcal{W}_2 \mathcal{Z}_3) \\ M^{-}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{W}_3) &= \operatorname{sgn}(\mathcal{Z}_1 I \mathcal{Z}_2)\, \operatorname{sgn}(\mathcal{Z}_1 \mathcal{W}_3)\, \operatorname{sgn}(\mathcal{Z}_2 \mathcal{W}_3) \end{aligned}$$

The full 3pt amplitude is the sum of these:

$$M(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{W}_3) = M^{-}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{W}_3) + \tilde{M}^{+}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{W}_3)$$

$\mathcal{N} = 4$ SYM

For $\mathcal{N} = 4$ SYM, BCFW becomes:

$$M(\mathcal{W}_i, \mathcal{Z}_j) = \sum_{L \cup R} \int \left[D^{3|4}\mathcal{W}\ D^{3|4}\mathcal{Z}\right]_{ij} M_L(\mathcal{W}_i, \mathcal{Z}) M_R(\mathcal{Z}_j, \mathcal{W})$$

where the measure is integrals and sgn's:

$$\begin{aligned}\left[D^{3|4}\mathcal{W}\ D^{3|4}\mathcal{Z}\right]_{ij} &= D^{3|4}\mathcal{W}\ D^{3|4}\mathcal{Z}\ \operatorname{sgn}(\mathcal{W}\mathcal{Z}) \\ &\quad \operatorname{sgn}(\mathcal{W}_i\mathcal{Z}_j)\operatorname{sgn}(\mathcal{W} I \mathcal{W}_i)\operatorname{sgn}(\mathcal{Z} I \mathcal{Z}_j)\end{aligned}$$

where $\int \frac{d\tau}{\tau} e^{i\tau \mathcal{W}_i \mathcal{Z}_j} = \operatorname{sgn}(\mathcal{W}_i \mathcal{Z}_j)$.

Hodges diagrams

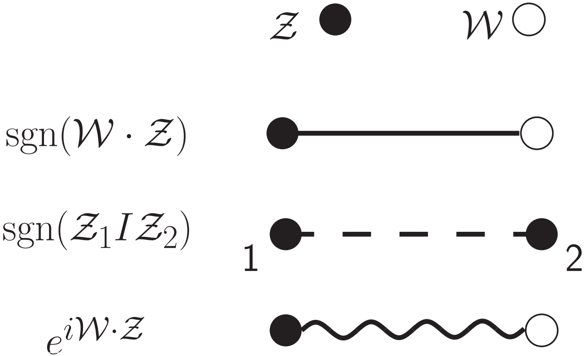
We see that in $\mathcal{N} = 4$ SYM, M_3 and BCFW consist solely of $\mathrm{sgn}()$'s and integrations over $D^{3|4}\mathcal{W}\ D^{3|4}\mathcal{Z}$.

There is a natural diagrammatic representation which has been studied for many years by Hodges.

In $(3,1)$, Hodges diagrams involve complex integrals with unknown contours. Not a problem in $(2,2)$.

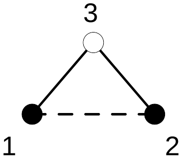
Most importantly, we will never have to do any actual integrals!

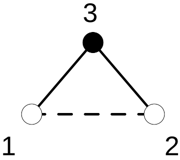
notation

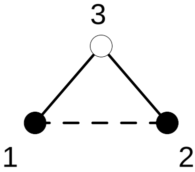


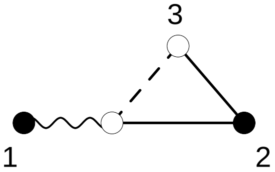
$\mathcal{Z}$
$\mathcal{W}$
$\mathrm{sgn}(\mathcal{W} \cdot \mathcal{Z})$
$\mathrm{sgn}(\mathcal{Z}_1 I \mathcal{Z}_2)$
1
2
$e^{i\mathcal{W}\cdot\mathcal{Z}}$

3pt

$M_3^- =$ 

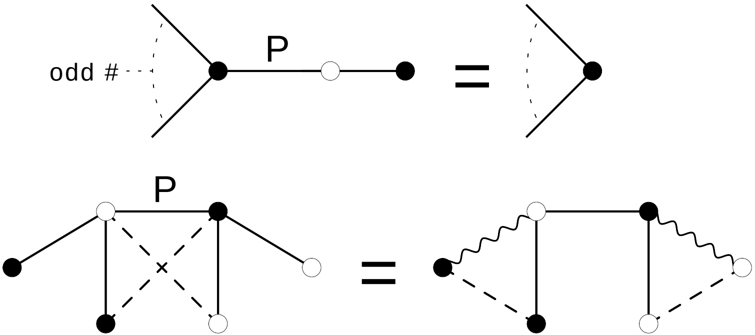
$M_3^+ =$ 

$M_3 =$ 

$+$ 

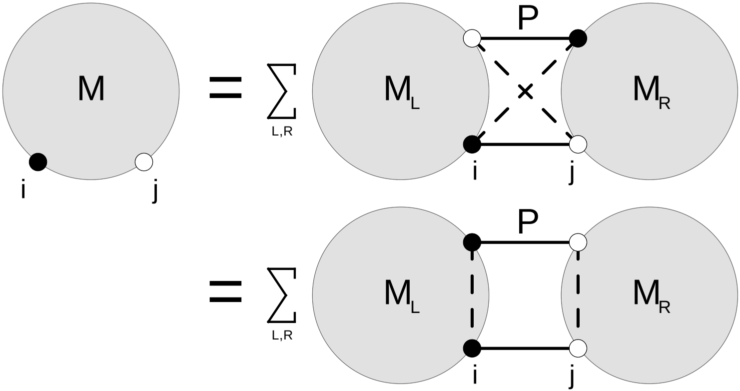
some identities

The "scrunch" and "butterfly" identities can all be proven straightforwardly in twistor space:



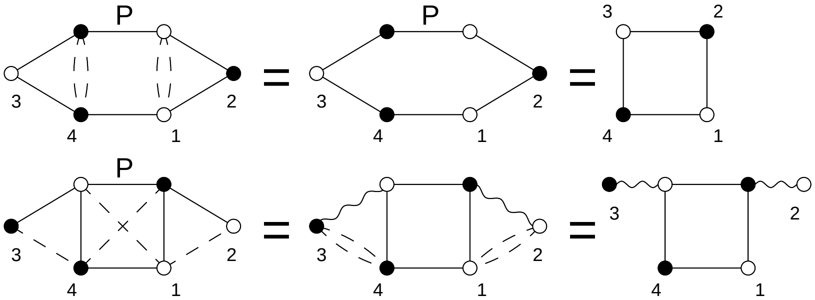
BCFW

Tree amplitudes in (S)YM take the form of disks:



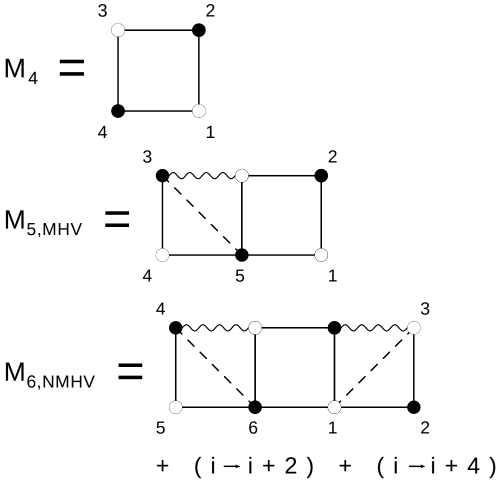
3pt $\longrightarrow$ 4pt

Two ways of computing the 4pt from BCFW:



Using $\mathrm{sgn}(x)^2 = 1$ is important.

4pt, 5pt, 6pt



These can all be written as products of M^+ or M^- triangles!

triangulations

If $M^{\pm}$ triangles are "letters," then let us define a "word" to be a product of such triangles.

Every $\mathcal{N} = 4$ SYM amplitude is a "sentence" given by a sum "words" in twistor space.

This is easily proven inductively.

But what does this mean in momentum space?

inverse soft factors

Multiplying by a triangle in twistor space means applying an "inverse soft factor" in momentum space.

So $M^+(123)M(13\ldots)$ in twistor space corresponds to adding a $(+)$ particle between 1 and 3 in momentum space:

$$G(1\,2^+3)\mathcal{M}(1\,3\ldots) = \frac{\langle 31\rangle}{\langle 12\rangle\langle 23\rangle}\mathcal{M}(1\,3\ldots)\Big|_{\tilde{\lambda}_{1,3}=\tilde{\lambda}'_{1,3}}$$

where $\tilde{\lambda}_{1,3}$ is shifted to conserve momentum:

$$\tilde{\lambda}'_1 = \frac{\langle 3|p_1+p_2|}{\langle 31\rangle}, \quad \tilde{\lambda}'_3 = \frac{\langle 1|p_2+p_3|}{\langle 13\rangle}$$

building the 6pt NMHV

One word (of three) in $\mathcal{M}(1^+2^-3^+4^-5^+6^-)$ is

$$\begin{aligned}
G(1\,2^-3)\,G(6\,1^+3)\,G(6\,3^+4)\mathcal{M}(4^-5^+6^-)\\
G(1\,2^-3)\,G(6\,1^+3)\mathcal{M}(3^+4^-5^+6^-)\\
G(1\,2^-3)\mathcal{M}(1^+3^+4^-5^+6^-)\\
\mathcal{M}(1^+2^-3^+4^-5^+6^-)
\end{aligned}$$

Let us show this explicitly.

building the 6pt NMHV

$$
\begin{aligned}
G(6\,3^+4)\mathcal{M}(4^-5^+6^-) &= \frac{\langle 64\rangle}{\langle 43\rangle\langle 36\rangle} \times \frac{\langle 64\rangle^3}{\langle 45\rangle\langle 56\rangle} \\
&= \frac{\langle 46\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 63\rangle}
\end{aligned}
$$

$$
\begin{aligned}
G(6\,1^+3)\mathcal{M}(3^+4^-5^+6^-) &= \frac{\langle 63\rangle}{\langle 31\rangle\langle 16\rangle} \times \frac{\langle 46\rangle^4}{\langle 45\rangle\langle 56\rangle\langle 63\rangle\langle 34\rangle} \\
&= \frac{\langle 46\rangle^4}{\langle 13\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}
\end{aligned}
$$

building the 6pt NMHV

$$G(1\,2^-3)\mathcal{M}(1^+3^+4^-5^+6^-) = \frac{[31]}{[12][23]} \times \frac{\langle 46\rangle^4}{\langle 1'3'\rangle\langle 3'4\rangle\langle 45\rangle\langle 56\rangle\langle 61'\rangle}$$

where the primed spinors are

$$|3'\rangle = \frac{[1|p_2+p_3|}{[13]}, \quad |1'\rangle = \frac{[3|p_1+p_2|}{[31]}$$

This gives the correct answer:

$$\frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle\langle 6|p_1+p_2|3]\langle 4|p_2+p_3|1](p_1+p_2+p_3)^2}$$

the rule?

Every $n$pt amplitude is a sentence of n-letter words.

Which words are allowed? Via identities, sentences can be translated into alternative forms.

Without resorting to BCFW, is there a "grammar"?

Number of BCFW terms = Catalan numbers.
Mapping to a combinatoric problem?

generating functionals

We can re-package the tree-level S-matrix of $\mathcal{N} = 4$ SYM into a convenient generating functional:

$$\mathbf{M}[\phi] = \sum_{n=3}^{\infty} \frac{1}{n!} \int D^{4|4}\mathcal{W}_1 \dots D^{4|4}\mathcal{W}_n \, \phi^{c_1}(\mathcal{W}_1) \dots \phi^{c_n}(\mathcal{W}_n) \, M^{c_1 \dots c_n}(\mathcal{W}_1 \dots \mathcal{W}_n)$$

The "propagator" in a general background is:

$$\mathbf{P}^{ab}[\phi](\mathcal{W}, \mathcal{Z}) = \frac{\delta^2 \mathbf{M}[\phi]}{\delta \phi^a(\mathcal{W}) \delta \tilde{\phi}^b(\mathcal{Z})}$$

$\star$ product $\equiv$ BCFW

Any two functionals $\mathbf{F}(\mathcal{W},\mathcal{Z})$ and $\mathbf{G}(\mathcal{W},\mathcal{Z})$ have a natural product in twistor space

$$(\mathbf{F}\star\mathbf{G})(\mathcal{W},\mathcal{Z}) = \int \left[D^{3|4}\mathcal{W}' D^{3|4}\mathcal{Z}'\right]_{\mathcal{W},\mathcal{Z}} \mathbf{F}(\mathcal{W},\mathcal{Z}')\mathbf{G}(\mathcal{W}',\mathcal{Z})$$

which is precisely the BCFW bridge. The 3pt amplitude can also be repackaged as

$$\mathbf{\Phi}(\mathcal{W},\mathcal{Z}) = \int d^{4|4}\mathcal{W}'\; M_3(\mathcal{W},\mathcal{Z},\mathcal{W}')\phi(\mathcal{W}')$$

holographic equation

Tree-level SYM and supergravity is reformulated as

$$\begin{aligned} \mathbf{P}^{ab} - \mathbf{P}^{ac} \star \mathbf{P}^{cb} &= g\, f^{ab}{}_{c}\, \mathbf{\Phi}^{c} \\ \mathbf{P} - \mathbf{P} \star \mathbf{P} &= \frac{1}{M_{\mathrm{Pl}}} \mathbf{\Phi} \end{aligned}$$

To extract M_n, simply apply $\frac{\delta^{n-2}}{\delta\phi(\mathcal{W}_3)\ldots\delta\phi(\mathcal{W}_n)}$.

Leibnitz rule does the BCFW partition!

Can we find new solutions to this equation? $\mathcal{O}(\hbar)$?

conclusions

- Twistor space is the natural home for the S-matrix. M_3 takes a striking form. BCFW reduces to $[D^4 Z \ D^4 W]_{ij}$.
- Tree amplitudes are simply computed and compactly represented using Hodges diagrams.
- There is evidence for a new rule that constructs the S-matrix from inverse soft factors.
- The tree-level dynamics of SYM and supergravity can be distilled into a holographic equation.

future directions

- find the “grammar” for amplitudes
- extend or solve the holographic equation
- explore the gravity S-matrix in twistor space
- better understand one-loop amplitudes in twistor space