Anomaly Inflow and topological mass terms
With Federico Bonetti, Ruben Minasian

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- Geometric Engineering of QFTs is a powerful tool for exploring Strongly Coupled Systems
- The Landscape of SCFTs can be explored by studying the low-energy dynamics of various brane systems
- Reduction of SCFTs on compact manifolds, $X$ – Lower D SCFT defined by $X$
- Typical SCFT is strongly coupled and may not admit Lagrangian descriptions [Gaiotto '09; Gaiotto, Moore, Neitzke '09]
- Many of such SCFTs can admit an arbitrarily large flavor symmetry – For example: Compactification of 6D SCFTs on punctured Riemann surfaces
- Physical observables of SCFTs from the geometric definitions

Compute 't Hooft anomalies of SCFTs from geometric setup
't Hooft Anomalies: Gauge anomalies for global symmetry $G$ – Poincaré, flavor, discrete, higher-form (p-form conserved currents) [Gaiotto, Kapustin, Seiberg, Willett, ’12]


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- **Quantum Anomalies**: Partition function not invariant under gauge transformation in presence of background gauge fields

\[
Z_{QFT}[A'] = e^{i\alpha(A,\epsilon)} Z_{QFT}[A]
\]

$A'$ & $A$ are background gauge fields of $G$ related by gauge transformation $\epsilon$

The anomaly measures obstruction to gauging the symmetry $G$

- Non-renormalized under RG flows
- Must be reproduced by any effective description
- Constrain IR phases of quantum systems
- Yield central charges in supersymmetric theories

't Hooft anomalies provide a measure for degrees of freedom for QFTs – Defining data for non-Lagrangian theories
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- $\alpha(A,\epsilon)$ cannot be removed by local counterterms $\Rightarrow$ Cohomology problem
- $\alpha(A,\epsilon)$ vanish in the limit $A \rightarrow 0 \Rightarrow$ Can imply higher groups otherwise
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't Hooft anomalies provide a measure for degrees of freedom for QFTs – Defining data for non-Lagrangian theories
Anomaly Polynomials

- Anomalies for **continuous** global symmetries
- The anomaly for a QFT on $W_d$ is given by an integral of a local density
  \[ \alpha(A, \epsilon) = \delta_\epsilon \mathcal{W}_{QFT}[A] = 2\pi \int_{W_d} I_d^{(1)} \]

- Wess-Zumino consistency conditions imply descent relations for anomaly
  \[ dl_d^{(1)} = \delta l_{d+1}^{(0)}, \quad dl_{d+1}^{(0)} = l_{d+2} \]

  - $l_{d+1}^{(0)}$ is a Chern-Simons form in $W_{d+1}$ with boundary $W_d$
  - $l_{d+2}$ is a gauge invariant form in $W_{d+2}$ with boundary $W_{d+1}$
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- $l_{d+2}$ is a gauge invariant form in $W_{d+2}$ with boundary $W_{d+1}$
- $l_{d+2}$ is a polynomial in curvatures of the background fields whose coefficients encode the 't Hooft anomaly of the global symmetry – **Anomaly Polynomial**
- Example in 4d: $a_{IJK}$ and $a_I$ are anomaly coefficients from triangle diagram

$$l_6 = a_{IJK} F^I \wedge F^J \wedge F^K + a_I F^I \wedge \text{tr}(R \wedge R),$$
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- $l_{d+2}$ is a **polynomial in curvatures of the background fields** whose coefficients encode the 't Hooft anomaly of the global symmetry – **Anomaly Polynomial**
- Captures anomalies for **Discrete Symmetries** when embedded in continuous symmetries
- Quantization conditions on background fields and anomaly polynomial – **Global anomalies** – Anomaly form as differential co-cycle
1. Anomalies of SCFTs in M-theory

2. Topological mass terms and discrete symmetry
Outline

1. Anomalies of SCFTs in M-theory

2. Topological mass terms and discrete symmetry
Setup with M5-branes

- Consider a stack of $N$ M5-branes in M-theory
  - Flat branes: $(2,0)\ A_{N-1}$ SCFTs in 6D
  - Probing $\mathbb{C}^2/\Gamma$ singularity: $(1,0)$ SCFTs in 6D
  - Wrapped on a surface $X$: SCFTs in 4D, SCFTs in 2D
  - ...
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- The 4-form flux of M-theory admits a singular magnetic source and the M-theory background has an internal boundary

$$dG_4 = N\delta W_6, \quad M_{11} = \mathbb{R}^+ \times M_{10}$$

- $M_{10}$ is the boundary of a tubular neighborhood of the source:
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$M_{10}$ is the boundary of a tubular neighborhood of the source:

$$M_{10-d} \hookrightarrow M_{10} \rightarrow W_d, \quad M_4 \hookrightarrow M_{10-d} \rightarrow X_{6-d}$$

- $M_{10-d}$: defines the SCFT in M-theory, can have orbifold fixed points
- $M_4$: The angular directions that surround the branes
- $M_4$ fibration fixed by topological twist

SCFT$_d$ on $W_d$ (external spacetime)
Symmetries and Anomalies

- Reducing M-theory on $M_{10-d}$ can lead to interesting gauge symmetry, $G$

- Components of $G$: the isometry group of $M_{10-d}$, massless fluctuations of the $C_3$ potential – Expanded on $H^*(M_{10-d}, \mathbb{Z})_{\text{free}}$

$$\delta C_3 = c_3 + b_2^u \lambda_1^u + a_1^\alpha \omega_\alpha^2 + t_0^x \Lambda_x^3$$

- Bulk gauge fields: $(c_3, b_2^u, a_1^\alpha, t_0^x) \rightarrow (2, 1, 0, (-1))$-form U(1) gauge symmetries
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- Due to the singular source of $G_4$, the classical variation of the M-theory action under diffeomorphisms and the gauge group $G$ is anomalous
- Consistency of the full theory, including the M5-brane sources, must be anomaly free

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**Anomaly Inflow**: The quantum anomalies for the boundary degrees of freedom on the M5-branes must cancel the classical bulk anomaly

The bulk supergravity action can be used to obtain the anomalies for SCFTs from M5-branes
The anomalous variation of the M-theory action depends on the boundary condition of $G_4$ corresponding to the singular source [Freed, Harvey, Minasian, Moore '98]

$$G_4 = 2\pi \rho(r) \bar{G}_4 + \cdots \quad \text{with} \quad \int_{M_4} \bar{G}_4 = N$$

$\rho(r)$ is a bump function that vanishes away from the boundary.
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$$G_4 = 2\pi \rho(r) \tilde{G}_4 + \cdots \quad \text{with} \quad \int_{M_4} \tilde{G}_4 = N$$

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The boundary term $\tilde{G}_4$ is a closed and globally defined four-form on $M_{10-d}$.

$\tilde{G}_4$ can be extended to a closed, gauge invariant and globally defined four-form, $E_4$, on the space $M_{10}$ by gauging the action of the group $G$

$$\tilde{G}_4 \quad \text{on} \quad M_{10-d} \implies E_4 \quad \text{on} \quad M_{10}$$
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$$\tilde{G}_4 \quad \text{on} \quad M_{10-d} \quad \Rightarrow \quad E_4 \quad \text{on} \quad M_{10}$$

On $W_d$, the gauging corresponds to turning on background fields for the global symmetry

Background fields $\iff$ Boundary value of bulk gauge fields
The variation of the M-theory action localizes on the boundary
\[ \frac{\delta S_M}{2\pi} = \int_{M_{10}} \mathcal{I}_{10}^{(1)}, \quad d\mathcal{I}_{10}^{(1)} = \delta \mathcal{I}_{11}^{(0)}, \quad d\mathcal{I}_{11}^{(0)} = \mathcal{I}_{12} \]

The 12-form anomaly polynomial is completely characterized by \( E_4 \) and the M-theory action
\[ \mathcal{I}_{12} = -\frac{1}{6} E_4 \wedge E_4 \wedge E_4 - E_4 \wedge X_8 \]

the 8-form, \( X_8 = \frac{1}{192} [p_1(TM_{11})^2 - 4p_2(TM_{11})] \sim R^4 \), decomposed on \( M_{11} = \mathbb{R}^+ \times M_{10} \) – Gravitational anomalies
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Anomaly inflow statement:

\[
i_{d+2}^{\text{inf}} + i_{d+2}^{\text{CFT}} + i_{d+2}^{\text{decoupled}} = 0, \quad i_{d+2}^{\text{inf}} = \int_{M_{10-d}} I_{12}
\]
Anomaly for 6D (2, 0) Theory – Flat branes

- $M_{10}$ and boundary condition for $G_4$ are

  \[ M_{10} = W_6 \times S^4, \quad \bar{G}_4 = N \, d\Omega_4 \]

$M_4$: Round 4-sphere and the induced global symmetry is $SO(5)$ – the R-symmetry of the (2, 0) SCFT
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The extension of $\tilde{G}_4$: global angular form of the 4-sphere

$$E_4 = \frac{N}{64\pi^2} \epsilon_{a_1 \cdots a_5} y^{a_5} \left[ D y^{a_1} \cdots D y^{a_4} + 2 F^{a_1 a_2} D y^{a_3} D y^{a_4} + F^{a_1 a_2} F^{a_3 a_4} \right]$$

$$D y^a = d y^a - A^{ab} y^b, \quad y^a y^a = 1$$

Here $A^{ab}$ is the $SO(5)$ connection with field strength $F^{ab}$
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- Integrating $\mathcal{I}_{12}$ on $S^4$: [Freed, Harvey, Minasian, Moore ’98; Harvey, Minasian, Moore ’98]
  \[
  I_{8}^{\text{inf}} + I_8[(2, 0) \text{ SCFT}] + I_8[\text{Free (2,0) tensor}] = 0
  \]
The extension $E_4$ has different components

$$E_4 = \sum_\mathcal{p} E_4^{\mathcal{p}}$$

$E_4^{\mathcal{p}}$: expansion along a basis of $H^p(M_{10-d}, \mathbb{Z})_{\text{free}}$

$$\tilde{\mathcal{G}}_4 = N^a \omega^4_a \rightarrow E_4^4 = N^a \left[ \Omega^4_{a} g + F^I \omega^g_{a,I} + F^I F^J \sigma_{a,IJ} \right]$$

$F^I = DA^I$: Background gauge fields for isometry group
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- Closure of $E_4$:
  $$\iota_I \Omega^4_a + d\omega_{a,I} = 0, \quad \iota_I (\iota_J \omega_{a,J}) + d\sigma_{a,IJ} = 0$$
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The expansion along 2-forms is

$$E_4^2 = F^\alpha \left[ \omega_{2,g}^\alpha + F^I \sigma_{\alpha,I} \right], \quad \iota_I \omega_2^\alpha + d\sigma_{\alpha,I} = 0$$

$F^\alpha = dA^\alpha$ – background field in QFT $\iff$ Boundary value of bulk gauge field $f^\alpha = da^\alpha$
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\( F^\alpha = dA^\alpha \) – background field in QFT \( \iff \) Boundary value of bulk gauge field \( f^\alpha = da^\alpha \)

**Choices for \( E_4 \) labeled by \( G_{\text{isom}} \)-equivariant cohomology of \( M_{10-d} \)**
Compute anomaly by considering local ansatz for metric and p-forms on $M_{10-d}$ consistent with symmetry and topology

- Impose regularity conditions on $E_4$
- Regularity conditions related to integrals of internal forms $(\Omega^4, \omega^2, \ldots)$
- The Inflow anomaly depends on background fields and on flux parameters of $M_{10-d}$
Consider an $\text{AdS}_{d+1} \times \mathcal{M}_{10-d}$ solution in M-theory supported by a $G_{ads}^4$ flux.

We can identify $\mathcal{M}_{10-d} = M_{10-d}$ and $G_{ads}^4 = \bar{G}_4$.

The 4-form $E_4$ can be constructed and $\mathcal{I}_{12}$ yields the anomaly for the dual SCFT.

The $X_8$ term in $\mathcal{I}_{12}$ yields the $\frac{1}{N^2}$ corrections to the anomaly polynomial.

Extremization principles [Intriligator, Wecht '03; Benini, Bobev '15]

We expect the anomaly to be exact up to $O(1)$ corrections due to decoupled center-of-mass degrees of freedom.
1 Anomalies of SCFTs in M-theory

2 Topological mass terms and discrete symmetry
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Example: consider an $M_6$ with closed p-forms, $(\lambda_{u}^{1}, \omega_{\alpha}^{2})$, one expects massless fluctuations for $C_{3}$ of the form

$$\delta C_{3} = a_{1}^{\alpha} \land \omega_{\alpha}^{2} + b_{2}^{u} \land \lambda_{u}^{1} + c_{3} + t_{0}^{x} \Lambda_{x}^{3}$$

$(a_{1}^{\alpha}, b_{2}^{u}, c_{3})$: gauge fields in 5D spacetime for U(1) $(0, 1, 2)$-form gauge symmetries.
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M-theory Chern-Simons can lead to topological mass terms of the 5D theory

$$\mathcal{L} = \frac{1}{2\pi} \Omega_{uv} b^u_2 \wedge db^v_2 + \frac{N_\alpha}{2\pi} a^\alpha_1 \wedge dc_3 + \cdots$$

Gauge symmetry is spontaneously broken – dual continuous global symmetry is not present.
In the reduction of M-theory on $M_{10-d}$, there can be **topological mass terms** and part of the gauge symmetry is **spontaneously broken**

Example: consider an $M_6$ with closed p-forms, $(\lambda^1_u, \omega^2_\alpha)$, one expects massless fluctuations for $C_3$ of the form

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**Topological mass terms have important consequences for anomaly inflow results!**
Consider the 5d topological action \[ S = \frac{M}{2\pi} \int_{\mathcal{M}_5} b_2 \wedge d\tilde{b}_2 + \frac{k}{2\pi} \int_{\mathcal{M}_5} c_3 \wedge da_1 \] 

Large gauge transformations imply that \((M, k)\) are integers

On Shell: \( dc_3 = da_1 = db_2 = d\tilde{b}_2 = 0 \) – no local operators
Consider the 5d topological action $[\text{Banks, Seiberg '11}]
\begin{align*}
S &= \frac{M}{2\pi} \int_{\mathcal{M}_5} b_2 \wedge d\tilde{b}_2 + \frac{k}{2\pi} \int_{\mathcal{M}_5} c_3 \wedge da_1 \\
\end{align*}

- Large gauge transformations imply that $(M, k)$ are integers
- On Shell: $dc_3 = da_1 = db_2 = d\tilde{b}_2 = 0$ – no local operators
- Observables: Holonomies of gauge fields – “Wilson lines”
\begin{align*}
W_c(C_3, n) &= \exp \left( i \ n \int_{C_3} c_3 \right), \\
W_a(C_1, n) &= \exp \left( i \ n \int_{C_1} a_1 \right) \\
W_b(C_2, n, \tilde{n}) &= \exp \left( i \int_{C_2} [\tilde{n} \ b_2 - n \ \tilde{b}_2] \right)
\end{align*}

Correlation functions of “Wilson lines” implies that $c_3, a_1$ are flat connections with holonomies in $\mathbb{Z}^k \in \mathbb{U}(1)$
$b_2$, $\tilde{b}_2$ are flat connections with holonomies in $\mathbb{Z}^M \in \mathbb{U}(1)$

Topological mass terms are dual to the Stückelberg action – Discrete symmetry left over from spontaneous breaking of $\mathbb{U}(1)$ symmetries
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Observables: Holonomies of gauge fields – “Wilson lines"

\[
W_c(C_3, n) = \exp \left( i \ n \int_{C_3} c_3 \right), \quad W_a(C_1, n) = \exp \left( i \ n \int_{C_1} a_1 \right)
\]

\[
W_b(C_2, n, \tilde{n}) = \exp \left( i \int_{C_2} [\tilde{n} \ b_2 - n \ \tilde{b}_2] \right)
\]

Correlation functions of “Wilson lines” implies that

- \(c_3, a_1\) are flat connections with holonomies in \(\mathbb{Z}_k \in U(1)\)
- \(b_2, \tilde{b}_2\) are flat connections with holonomies in \(\mathbb{Z}_M \in U(1)\)

Topological mass terms are dual to the \textbf{Stückelberg action} – Discrete symmetry left over from spontaneous breaking of \(U(1)\) symmetries
In suitable normalization of gauge fields, and due to flux quantization, \((\Omega_{uv}, N_\alpha)\) are quantized.

The topological mass terms describe discrete gauge symmetries in the 5D supergravity.

For \(\Omega_{12} = M\), and \(k = \gcd(N_\alpha)\) the discrete gauge symmetries are:

\[
\begin{align*}
\mathbb{Z}_k & \quad \text{2-form with} \quad c_3 \\
\mathbb{Z}_k & \quad \text{0-form with} \quad a_1 = m_\alpha a_1^\alpha, \quad N_\alpha = k m_\alpha \\
\mathbb{Z}_M \times \mathbb{Z}_M & \quad \text{1-form with} \quad (b_2^1, b_2^2)
\end{align*}
\]
The boundary global symmetry dual to the discrete gauge symmetry depends on the choice of boundary condition for the gauge fields [Witten '99]

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- Dirichlet boundary conditions fix a source for discrete symmetry in the dual theory [Gaiotto, Kapustin, Seiberg, Willett '14; Hofman, Iqbal, '18]

- Mixed boundary conditions between the fields lead to a larger class of possible choices of boundary discrete symmetry [Gaiotto, Kapustin, Seiberg, Willett '14]

- Similar choices exist for the 1-form discrete symmetry from $(b_2, \tilde{b}_2)$
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In case (c), there is a **mixed 't Hooft anomaly** between the two discrete symmetry

[Gaiotto, Kapustin, Seiberg, Willett '14; Bergman, Tachikawa, Zafrir '20]
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  [Gaiotto, Kapustin, Seiberg, Willett ’14; Bergman, Tachikawa, Zafrir ’20]

- Formally the anomaly polynomial includes
  $$I_6 \supset k \frac{dA_1}{2\pi} \wedge \frac{d\tilde{C}_3}{2\pi} + \Omega_{uv} \frac{dB_2^u}{2\pi} \wedge \frac{dB_2^v}{2\pi}$$

- $(A_1, \tilde{C}_3, B_2^u, B_2^v)$ are the boundary values of the gauge field $(a_1, c_3, b_2^u, b_2^v)$
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- \((A_1, \tilde{C}_3, B_{2u}^v, B_{2v}^u)\) are the boundary values of the gauge field \((a_1, c_3, b_{2u}^v, b_{2v}^u)\)
- These anomalies determine the surface and line operators that can exist for the gauge theory
- From the bulk, the choice of boundary condition determines which bulk “Wilson lines” can end on the boundary
In general, the anomaly polynomial includes terms

\[ I_6 \supset N_\alpha \frac{F^\alpha}{2\pi} \wedge \frac{d\tilde{C}_3}{2\pi} + \mathcal{K}_\alpha \cdot \frac{F^\alpha}{2\pi} \wedge Q^\bullet_4 + \mathcal{K} \cdot \frac{d\tilde{C}_3}{2\pi} \wedge \tilde{Q}^\bullet_2 \\
+ \mathcal{K}_{\alpha\beta} \cdot \frac{F^\alpha}{2\pi} \wedge \frac{F^\beta}{2\pi} \wedge Q^\bullet_2 + \mathcal{K}_{\alpha\beta\gamma} \frac{F^\alpha}{2\pi} \wedge \frac{F^\beta}{2\pi} \wedge \frac{F^\gamma}{2\pi} \]

The $\mathcal{K}$’s are intersection numbers from various 2-forms in $M_6$, $F^\alpha = dA^\alpha$

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Disclaimer: Anomaly polynomial and its component forms should be understood as field strength for differential co-cycles
Mixed ’t Hooft Anomalies

In general, the anomaly polynomial includes terms

\[ I_6 \supset N_\alpha \frac{F^\alpha}{2\pi} \wedge \frac{d \tilde{C}_3}{2\pi} + K_\alpha \bullet \frac{F^\alpha}{2\pi} \wedge Q_4^\bullet + K_\bullet \frac{d \tilde{C}_3}{2\pi} \wedge \tilde{Q}_2^\bullet + K_{\alpha\beta} \frac{F^\alpha}{2\pi} \wedge \frac{F^\beta}{2\pi} \wedge Q_2^\bullet + K_{\alpha\beta\gamma} \frac{F^\alpha}{2\pi} \wedge \frac{F^\beta}{2\pi} \wedge \frac{F^\gamma}{2\pi} \]

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A basis transformation \((A^\alpha) \longrightarrow (A_1, A_\alpha')\) that is consistent with quantization of flux is necessary

When successful mixed ’t Hooft anomalies between discrete and continuous symmetries can be read off from the anomaly polynomial
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\[ I_6 \supset N_\alpha \frac{F^\alpha}{2\pi} \wedge \frac{d\tilde{C}_3}{2\pi} + K_{\alpha \bullet} \frac{F^\alpha}{2\pi} \wedge Q_4^\bullet + K_\bullet \frac{d\tilde{C}_3}{2\pi} \wedge \tilde{Q}_2^\bullet \]

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If we care only about perturbative anomalies, things are less subtle: Impose equation of motion for massive bulk fields on background fields

\[ N_\alpha F^\alpha + K_\bullet \tilde{Q}_2^\bullet = 0 \]
Anomalies for continuous symmetries

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- \( \tilde{Q}^\bullet_2 \) is either field strength for a 0-form symmetry or products of \( dT_0^x \) – Boundary field strength for axions

- \( T_0^x \) – background dependent coupling parameters!
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\[ T_0^x \text{ – background dependent coupling parameters!} \]

- Constraints on background fields translate to constraints on symmetry generators \( J^\alpha \):

\[ N_\alpha F^\alpha \rightarrow M_\alpha J^\alpha = 0 \]

a-maximization for CFT is sensitive to constraints over U(1) symmetries that can mix with the R-symmetry
Green-Schwarz terms [IB, Bonetti, Minasian, Nardoni '19]

- Constraint on anomalies for continuous symmetries in 6d
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- Constraint on anomalies for continuous symmetries in 6d

\[ I_8 \rightarrow I_8 + \frac{1}{4} \frac{d\tilde{C}_3}{2\pi} \wedge \frac{d\tilde{C}_3}{2\pi} + \frac{d\tilde{C}_3}{2\pi} Q_4 \rightarrow I_8 - Q_4^2 \]
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- **Anomaly inflow for 6D (1, 0) SCFTs from M5 branes at orbifolds** [Ohmori, Shimizu, Tachikawa, Yonekura, ’14]

  Interpreted as a Green-Schwarz term associated to the decoupled center of mass mode of the stack in Ohmori et al.
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- Bulk equation of motion fix Green-Schwarz term!
In presence of a boundary, BF theories admit singleton modes [Witten '99; Maldacena, Moore, Seiberg '01]

Singletons: Pure gauge modes in the bulk and dynamical in the boundary

\[ \frac{M}{2\pi} b_p \wedge da_{d-p-1} \rightarrow (p-1)\text{-form gauge field singleton} \]

SUSY partners from KK singletons
Singletons and Decoupled modes [IB, Bonetti, Minasian: 2007.15003]

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  I^{\text{inf}} + I^{\text{CFT}} + I^{\text{decoupled}} = 0
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- Singletons account for all decoupling modes in SUSY compactifications of M5-branes on punctured Riemann surfaces! (not including orbifold theories)

The symmetry and topology of \(M_{10-d}\) completely fix the anomaly of SCFTs from M5-branes and its compactifications
Applications to orbifold theories

- Consider a stack of $N$ $M5$-branes wrapped on a Riemann surface $\Sigma_g$ and probing a $\mathbb{C}^2/\mathbb{Z}_k$ singularity.
- The linking space $M_4 = S^4/\mathbb{Z}_k$, there are two $\mathbb{Z}_k$ orbifold fixed points at the poles.
- Space that define the QFT is $M_6 = M_4 \times \Sigma_g$ with a topological twist to preserve SUSY.
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There is an additional twist parameter from the $U(1)$ commutant of the R-symmetry in the isometry group of $S^4$. 
Symmetry of system

One-cycles:

\[ [\lambda^u, \tilde{\lambda}^u] \text{ on the } \Sigma_g, \ b^1(M_6) = 2g \]

- \( U(1)^{2g} \) 1-form gauge symmetry with \( Sp(2g, \mathbb{Z}) \) S-duality group
- Topological mass terms break gauge symmetry to \( (\mathbb{Z}_N \times \mathbb{Z}_N)^g \) 1-form symmetry
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Two-cycles:

\[ C_2^\alpha = (C_i^j, \Sigma_g) \] on \( M_6 \), \( b^2(M_6) = 2k - 1 \)

- \( U(1)^{2k-1} \) 0-form gauge symmetry
- Since \( b^0(M_6) = 1 \), Topological mass term involving a linear combination \( N_\alpha a_1^\alpha \wedge c_3 \)
- There is \( \mathbb{Z}_k \) 2-form and \( U(1)^{2(k-1)} \times \mathbb{Z}_k \) 0-form gauge symmetry, \( k = \gcd(N_\alpha) \)

[\text{C´ordova, Freed, Lam, Seiberg, '19}]
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Three-cycles:
\( (\lambda^u \times C_2^i, \tilde{\lambda}^u \times C_2^i) \) on the \( \Sigma_g \), \( b^3(M_6) = 4g(k - 1) \)
- \( 4g(k - 1) \) bulk axions, Boundary value of axions correspond to marginal coupling parameters
- Anomaly involving the axions correspond to anomalies in the space of couplings

[Córdova, Freed, Lam, Seiberg, '19]
Anomalies for $k = 1, 2$ were studied [IB, Bonetti, Minasian '20], For $k > 2$, To appear!
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Discrete symmetries and higher-form symmetries – role of **torsion in Cohomology group**

Anomalies related to large gauge transformations and duality groups of QFTs – Global anomalies

Defects and extended operators – higher-form discrete symmetry
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- Discrete symmetries and higher-form symmetries – role of torsion in Cohomology group
- Anomalies related to large gauge transformations and duality groups of QFTs – Global anomalies
- Defects and extended operators – higher-form discrete symmetry
- Explore general compactifications of 6D theories in IIB/F-theory (Inflow polynomial in [IB, Bonetti, Minasian, Weck ’20]), massive IIA
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- Explore general compactifications of 6D theories in IIB/F-theory (Inflow polynomial in [IB, Bonetti, Minasian, Weck ’20]), massive IIA
- Conformal blocks relating to Singleton physics and anomalies relating to $Sp(2g, \mathbb{Z})$ duality group (Similar to [Belov, Moore ’04] )
Outlook

- Anomalies for $k = 1, 2$ were studied [IB, Bonetti, Minasian ’20], For $k > 2$, To appear!
- Discrete symmetries and higher-form symmetries – role of torsion in Cohomology group
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- Since the analysis relies less on SUSY, we hope to be able to study more general classes of compactifications with punctures and defects
- Topological mass terms in 5d supergravity encode discrete global symmetries of the dual field theory
- The same bulk theory with different topological boundary conditions gives field theories with different discrete global symmetries
Summary

- Topological mass terms in 5d supergravity encode discrete global symmetries of the dual field theory
- The same bulk theory with different topological boundary conditions gives field theories with different discrete global symmetries
- We can capture ‘t Hooft anomalies with a 6-form inflow anomaly polynomial
- There is a rich interplay between all p-forms fields from expansion of M-theory $C_3$ potential
  - Higher-form symmetries
  - Discrete symmetries
  - Anomalies in the space of coupling constants, or “(–1)-form” symmetries
THANK YOU!
One can also consider brane systems in type II string theories

The polynomials that encode the anomalies are 11-forms, $\mathcal{I}_{11}$ constructed from gauge invariant boundary conditions of various flux

The anomaly polynomial of IIA is related to the M-theory $\mathcal{I}_{12}$ by a reduction, It is similarly characterized by IIA Chern-Simons terms

The anomaly polynomial for IIB receives a contribution from the kinetic term of the self-dual five-form flux

If we consider a stack of D3-branes supported by the five-form flux, $F_5$

$$F_5 = 2\pi(1 + \star)\rho(r)\bar{F}_5 + \cdots \text{ on } M_{10} = \mathbb{R}^+ \times W_d \times M_{9-2d}$$

The boundary term $\bar{F}_5$ on $M_{9-2d}$ can be extended to $E_5$ on $W_d \times M_{9-2d}$

The 11-form and the inflow anomaly polynomial are given as

$$\mathcal{I}_{11} = \frac{1}{2} E_5 \wedge dE_5 - E_5 \wedge H_3 \wedge F_3, \quad I_{2d+2}^{\text{inf}} = \int_{M_{9-2d}} \mathcal{I}_{11}$$
The 11-form and the inflow anomaly polynomial are given as
\[ I_{11} = \frac{1}{2} E_5 \wedge dE_5 - E_5 \wedge H_3 \wedge F_3, \quad I_{2d+2}^{\text{inf}} = \int_{M_{9-2d}} I_{11} \]

For \( \mathcal{N} = 4 \) SYM, \( E_5 \) is the global angular form of the 5-sphere, \( e_5 \). Integrating \( I_{11} \) yields the anomaly for the \( SO(6) \) R-symmetry group
\[ E_5 = N e_5, \quad dE_5 = -N \pi^* \chi(SO(6)), \]
\[ I_{6}^{\text{inf}} = \frac{1}{2} N^2 \chi(SO(6)) = \frac{1}{2} N^2 c_3(SU(4)) \]

For more general \( \mathcal{N} = 1 \), \( E_5 \) is the volume of \( SE_5 \) gauged over the world volume theory! Consistent with holographic analysis by [Benvenuti, Pando Zayas, Tachikawa 06]

Anomaly of \( \mathcal{N} = 4 \) SYM on punctured Riemann surface

This anomaly formula can be used to study compactifications of 4D SCFTs to 2D QFTs
Generalize type IIB with non-trivial axio-dilaton profile

Consider an elliptic fibration over the IIB background

$$E_\tau \hookrightarrow M_{12} \to M_{10}$$

The anomaly polynomial is

$$\mathcal{I}_{11} = \frac{1}{2} E_5 \wedge dE_5 - E_5 \wedge \pi_* \left[ X_8(TM_{12}) + \frac{1}{2} \mathcal{E}_4 \wedge \mathcal{E}_4 \right]$$

$F_3$ and $H_3$ are encoded in $\mathcal{E}_4$, for trivial elliptic fiber

$$\mathcal{E}_4 = F_3 \wedge dx + H_3 \wedge dy$$

Anomalies of $\mathcal{N} = 4$ with varying coupling, $\tau_{YM}$, can be studied with this generalization [Lawrie, Martelli, Schäfer-Nameki '18]
Things to do

- Compute the anomalies for $\mathcal{N} = 2$ Class $S$ of $A_N$ type with arbitrary punctures [IB, Nardoni, ’18; IB, Bonetti, Minasian, Nardoni ’19]

- The possible choices of $E_4$ from $M_6 = S^4 \times \Sigma_{g,n}$ is in one-to-one correspondence with the classification from Hitchin equations

- Choices come from different resolutions of punctures on $\Sigma_{g,n}$ in $M_6$

- This provides an alternate derivation of punctures and the data associated with them from bulk SUGRA

- Explore punctures for $\mathcal{N} = 1$ Class $S$ [IB, Beem, Bobev, Wecht ’12] and from Class $S_k$ [Gaiotto, Razamat, ’15; Hanany, Maruyoshi ’15 and $S_\Gamma$ [Heckmann, Jefferson, Rudelius, Vafa, ’16]

- Study Class $S$ from the $D$-series (Inflow for 6D SCFT from [Yi, ’00]) and E-string theories

- Example – Class $S_2$
Consider a stack of $N$ M5-branes on $\Sigma_g$ and probing a $\mathbb{Z}_2$ orbifold fixed point

Here $M_6 = M_4 \times \Sigma_g$ and $M_4$ is $S^4/\mathbb{Z}_2$ with resolution two cycles

The resolution is supported by threading flux $(N^N, N^S)$ on 4-cycles made from the resolution 2-cycles combined with the Riemann surface

There are a total of three 4-cycles with three flux parameters $(N, N^N, N^S)$, Associated to them are three closed 2-forms by Poincare duality

The isometry group is $U(1)_R \times SU(2)_F$ and the naive symmetry from $C_3$ is $U(1)^3$

From the 6d $(1, 0)$ theory, only $U(1)_N \times U(1)_S$ is visible, the third $U(1)_C$ is an accidental symmetry from the compactification!
A combination of the three $U(1)$s is broken by a topological mass – **Spontaneous symmetry break of a $U(1)$ global symmetry** for the field theory

The symmetry of low-energy theory is then $U(1)_N' \times U(1)_S' \times U(1)_R \times SU(2)_L$

The generators of the 2 $U(1)$s visible from the 6d SCFT are shifted as

$$T'_N = T_N - \frac{N^N}{N} T_C, \quad T'_S = T_S - \frac{N^S}{N} T_C$$

After obtaining anomaly polynomial, compute large $N$ central charge by a-maximization [Intriligator, Wecht '03]

Inflow data can be matched with a family of $AdS_5 \times \mathcal{M}_6$ obtained in [Gauntlett, Martelli, Sparks, Waldram '04]
5d SUGRA theory admits a rich discrete gauge symmetry! Thus complex network of discrete symmetry in SCFT which is acted upon by $Sp(2g, \mathbb{Z})$

<table>
<thead>
<tr>
<th>multiplicity</th>
<th>fields</th>
<th>top. mass terms</th>
<th>bulk gauge symm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^2(M_6) = 3$</td>
<td>$a_1^a$</td>
<td>$\frac{1}{2\pi} N_a a_1^a \wedge dc_3$</td>
<td>$U(1)^2$ 0-form symm.</td>
</tr>
<tr>
<td>1</td>
<td>$c_3$</td>
<td></td>
<td>$\mathbb{Z}_k$ 0-form symm.</td>
</tr>
<tr>
<td>$b^1(M_6) = 2g$</td>
<td>$b_2^i, \tilde{b}_2^i$</td>
<td>$\frac{1}{2\pi} M \tilde{b}_2^i \wedge db_2^i$</td>
<td>$(\mathbb{Z}_M \times \mathbb{Z}_M)^g$ 1-form symm.</td>
</tr>
<tr>
<td>$b^3(M_6) = 4g$</td>
<td>$a_0^{i\pm}, \tilde{a}_0^{i\pm}$</td>
<td>—</td>
<td>5D axions</td>
</tr>
</tbody>
</table>

There are $4g$ background 1-forms in the anomaly polynomial associated to the axions – Anomaly for background dependent couplings and “(-1)-form symmetry”? [Córdova, Freed, Lam, Seiberg, ’19]
Origin of decoupled modes from $M_{10-d}$

\[ i_{\text{inf}} + i_{\text{QFT}} + i_{\text{decoupled}} = 0 \]

- Discrete symmetries and higher form symmetries – role of torsion in Cohomology group
- Anomalies related to large gauge transformations and duality groups of QFTs – Global anomalies
- Defects and extended operators – higher form discrete symmetry
- Explore general compactifications of 6D theories in IIB/F-theory (Inflow polynomial in [IB, Bonetti, Minasian, Weck '20]), massive IIA

Since the analysis relies less on SUSY, we hope to be able to study more general classes of compactifications with punctures and defects
THANK YOU!
When the stack of M5-branes is probing a $\mathbb{C}^2/\mathbb{Z}_k$ fixed point, $M_4 \cong S^4/\mathbb{Z}_k$

$\mathbb{Z}_k \subset SU(2)_L$ from $SU(2)_L \times SU(2)_R \subset SO(5)$ of the isometry group

When $k = 2$, the orbifold action preserves the $SU(2)_L \times SU(2)_R$ subgroup

On the branes, $SU(2)_L$ is a flavor symmetry and $SU(2)_R$ is an R-symmetry for the worldvolume $(1, 0)$ SCFT

There are two $\mathbb{R}^4/\mathbb{Z}_2$ fixed points on the sphere at the north and south poles

The fluctuations of the $C_3$ potential leads to an additional $SU(2)_N \times SU(2)_S$ flavor symmetry for the worldvolume theory

For the purpose of the SUGRA analysis, we consider a resolution of the orbifold fixed points by blowing up two-cycles at the poles of the sphere

Symmetry breaks:

$SU(2)_N \times SU(2)_S \times SU(2)_R \times SU(2)_L \rightarrow U(1)_N \times U(1)_S \times U(1)_R \times SU(2)_L$
The space $M_4$ is a circle fibration, $S^1_\psi$, over a cylinder $[\mu] \times S^2_\varphi$. The isometries of $S^1_\psi \times S^2_\varphi$ correspond to $U(1)_R \times SU_L(2)$

$S^1_\psi \times S^2_\varphi$ have a topology of $S^3/\mathbb{Z}_2$

The circle $S^1_\psi$ shrinks at the end points of the $\mu$-interval while the two sphere $S^2_\varphi$ never shrinks

The non-shrinking sphere at the end of the $\mu$-interval correspond to the blowup two-cycles of the orbifold fixed points
Now we consider the case when the branes wrap a Riemann surface $\Sigma_g$ while probing the singularity.

This is equivalent to taking the 6D $(1,0)$ theory on a Riemann surface with a topological twist to preserve supersymmetry.

By anomaly matching, the anomaly of the 4D theories can be computed as

$$I_6 = \int_{\Sigma_g} I_8$$

Anomaly polynomial does not yield correct central charge for “potential” dual holographic solution.

Possible accidental symmetry and interesting decoupled modes!
In this case, $M_6 = M_4 \times \Sigma_g$. the R-symmetry circle, $S^1_{\psi}$, is twisted over the Riemann Surface with curvature $2(g - 1)$.

$M_6$ has three 4-cycles, two of them correspond to taking the product of the polar two-cycles of $M_4$ with $\Sigma_g$. The third is the embedding of $M_4$ in $M_6$.

Threading flux on these cycles yields three quantum number $(N, N^N, N^S)$.

There are three closed 2-forms dual to the 4-cycles. The vector fluctuations of $C_3$ along these forms implies three $U(1)$ gauge fields in the bulk supergravity.

This suggests a $U(1)^3$ flavor symmetry for the 4d theory.

Compactification of the 6D $(1,0)$ theory only sees $U(1)_N \times U(1)_S$; the third $U(1)_C$ is an accidental symmetry!
In the reduction of M-theory on $M_6$, a combination of the vectors from $C_3$ acquires a topological mass term from M-theory CS term

$$S_{5d} \supset N^\alpha \int \gamma_3 \wedge da_\alpha, \quad C_3 \supset a_\alpha \wedge \omega_\alpha + \gamma_3$$

This topological mass term can dualized to St"uckelberg kinetic term with $N^\alpha a_\alpha$ eating the axion dual to $\gamma_3$

In the bulk supergravity this is spontaneous breaking of a $U(1)$ gauge symmetry and on the boundary, it corresponds to spontaneous breaking of a $U(1)$ global symmetry!

The symmetry of low-energy theory is then $U(1)_N' \times U(1)_S' \times U(1)_R \times SU(2)_L$

The generators are shifted as

$$T'_N = T_N - \frac{N^N}{N} T_C, \quad T'_S = T_S - \frac{N^S}{N} T_C$$
We write the 4-form as
\[
E_4 = N (\mathcal{V}_0^g + \cdots) + N^N (\mathcal{V}_N^g + \cdots) + N^S (\mathcal{V}_S^g + \cdots)
+ F^0 (\omega_0^g + \cdots) + F_{4d}^N (\omega_N^g + \cdots) + F_{4d}^S (\omega_S^g + \cdots)
\]

The field strength for the vector fluctuations of $C_3$ are $(F^0, F_{4d}^N, F_{4d}^S)$, one of them is removed by the constraint
\[
NF^0 + N^N F_{4d}^N + N^S F_{4d}^S = 0
\]

This constraint also follows from the tadpole condition

The 4d curvatures are related to the 6d curvatures as
\[
F^N = N^N V_\Sigma + F_{4d}^N, \quad F^S = N^S V_\Sigma + F_{4d}^S
\]

The flux $(N^N, N^S)$ are background flux for the 6D flavor symmetry on the Riemann surface

\[
l_{6, \text{large } N}^{\text{infl}} = \frac{1}{(2\pi)^3} \left[ \frac{1}{2} N (\chi N - N^N + N^S) F_R^2 (F_N + F_S) - \frac{1}{2} (N^N - N^S) F_R (F_N + F_S)^2 \right.
\]
\[
+ N^{-1} (N^N F_N + N^S F_S) (F_N^2 - F_S^2) - \frac{2}{3} \chi (F_N^3 + F_S^3) \right]
\]
To check for the existence of a SCFT fixed point, we look for an $AdS$ solution of the form

$$ds^2 = e^{2\lambda} \left[ ds^2(AdS_5) + e^{-6\lambda} ds^2(\tilde{M}_6) \right]$$

The solutions were already found by Gauntlett, Martelli, Sparks and Waldram in 2004!

By construction, symmetries and topology match

From our anomaly computation we can match the large $N$ central charge with $a$-maximization!

Class $S_2$ with a torus is dual to the $AdS_5 \times Y^{p,q}$ solutions in IIB supergravity