

Superpotentials and A_∞ Algebras

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Suppose we have a $(2d + 3)$ -brane of a type IIB string theory wrapped N -times around a complex d -dimensional subspace of a Calabi–Yau threefold X filling spacetime.

This gives rise to four-dimensional $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory in the worldvolume.

What is the superpotential, W ?

It can be computed purely from a knowledge of the geometry of X and the wrapped cycle (almost) entirely using topological field theory.

Similarly, for a suitable value of $B + iJ$, a D-brane might be marginally stable with respect to a decay into N_1 copies of some D-brane, N_2 copies of some other D-brane, etc.

This gives an $\mathcal{N} = 1$ gauge theory with gauge group $U(N_1) \times U(N_2) \times \dots$

The fact that the given D-brane is marginally stable means that there will be massless open strings between the decay products. These strings give rise to massless chiral supermultiplets in bifundamental $(\overline{\mathbf{N}}_1, \mathbf{N}_2)$ representations etc.

A “quiver” gauge theory – also with a superpotential.

Various techniques have been given in the literature for various example:

- Klebanov, Witten
- Morrison, Plesser
- Kachru, Katz, Lawrence, McGreevy
- Cachazo, Katz, Vafa
- Douglas, Govindarayan, Jayaraman, Tomasiello

Here we give a *general* solution to the problem (for $g_s \rightarrow 0$).

The key structure to the superpotential is an A_∞ -algebra (for one type of D-brane), or its generalization to an A_∞ -category (in the case of the quiver gauge theory).

Let V be a vector space with a \mathbb{Z} -grading and let $T(V)$ be the resulting graded tensor algebra

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}. \quad (1)$$

Now let \mathbf{d} be a derivative with degree 1, with respect to the grading, acting on $T(V)$ obeying the graded Leibniz rule

$$\mathbf{d}(a \otimes b) = \mathbf{d}(a) \otimes b + (-1)^a a \otimes \mathbf{d}(b). \quad (2)$$

We also demand

$$\mathbf{d}^2 = 0. \quad (3)$$

The Leibniz rule (2) means that \mathbf{d} is entirely determined by its restriction to V . Let us denote this restriction as $(\mathbf{d})_V$. One can then decompose

$$(\mathbf{d})_V = d_1 + d_2 + \dots, \quad (4)$$

where

$$d_k : V \rightarrow V^{\otimes k}. \quad (5)$$

Let $V[1]$ denote the vector space V with all the grades decreased by one and let $s : V \rightarrow V[1]$ be the obvious map of degree -1 . We can now define our A_∞ algebra A :

$$A = (V[1])^*. \quad (6)$$

together with its “higher products”

$$m_k : A^{\otimes k} \rightarrow A, \quad (7)$$

given by the dual of $s^{\otimes k} \cdot d_k \cdot s^{-1}$. The map m_k thus has degree $2 - k$.

The condition $\mathbf{d}^2 = 0$ then becomes equivalent to

$$\sum_{r+s+t=n} (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0, \quad (8)$$

for any $n > 0$, where $u = n + 1 - s$.

One may view (8) as the defining relations for an A_∞ algebra.

The A_∞ category may be defined similarly by giving the morphisms a higher product structure.

Now consider the topological B-model for open strings between B-type D-branes.

For example, for a single 9-brane wrapping X to form a vector bundle E , the open strings are given by Dolbeault cohomology $H_{\bar{\partial}}^{0,q}(X, \text{End}(E))$.

More generally D -branes are objects of the derived category $\mathbf{D}(X)$ and open strings are morphisms with Hilbert space $\text{Hom}(A, B[q]) = \text{Ext}^q(A, B)$.

The open strings are associated to local vertex operators ψ_i in the topological field theory.

These ψ_i 's may be viewed as a basis for A .

To each such vertex operator, one may construct a 1-form operator

$$\psi_i^{(1)} = \frac{1}{\sqrt{2}} \left\{ G_{-\frac{1}{2}}^- + \overline{G}_{-\frac{1}{2}}^-, \psi_i \right\}, \quad (9)$$

These 1-form operators may be used to deform the topological field theory (at least to first order):

$$S \rightarrow S + \sum_i Z_i \psi_i^{(1)}. \quad (10)$$

The Z_i are complex numbers as far as the topological field theory is concerned.

The Z_i (for $q = 1$) are (the scalar components of) chiral superfields in the effective world-volume theory.

The above deformations correspond to giving vacuum expectations values to these fields.

The chiral superfields are naturally dual to the vertex operators of the topological quantum field theory.

Define (up to suppressed signs)

$$B_{i_0, i_1, \dots, i_k} = \langle \psi_{i_0} \psi_{i_1} P \int \psi_{i_2}^{(1)} \int \psi_{i_3}^{(1)} \dots \int \psi_{i_{k-1}}^{(1)} \psi_{i_k} \rangle, \quad (11)$$

In the case of N copies of a simple D-brane, the fields Z_i naturally form $N \times N$ matrices. We may now write the superpotential

$$\mathbf{W} = \text{Tr} \left(\sum_{k=2}^{\infty} \sum_{i_0, i_1, \dots, i_k} \frac{B_{i_0, i_1, \dots, i_k}}{k+1} Z_{i_0} Z_{i_1} \dots Z_{i_k} \right).$$

If X is a Calabi–Yau threefold, there is also a “trace map” of degree -3

$$\gamma : A \rightarrow \mathbb{C}. \quad (12)$$

The correlation function may be written in the form

$$B_{i_0, i_1, \dots, i_k} = \gamma \left(m_2 \left(m_k(\psi_{i_0}, \psi_{i_1}, \dots, \psi_{i_{k-1}}), \psi_{i_k} \right) \right), \quad (13)$$

for maps of degree $2 - k$

$$m_k : A^{\otimes k} \rightarrow A. \quad (14)$$

It was shown by HLL that these products do indeed obey the conditions (8) and thus give A the structure of an A_∞ algebra.

There are two extreme versions of an A_∞ -algebra.

The first is where all the higher products vanish:

$$m_k = 0, \quad k > 2.$$

This is a differential graded algebra (dga).

m_1 is the differential and m_2 is the product.

The second is a minimal model, where $m_1 = 0$ but all higher products can be nonzero.

An A_∞ morphism is a set of maps

$$f_k : A^{\otimes k} \rightarrow B. \quad (15)$$

such that

$$\sum_{r+s+t=n} f_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum_{\substack{1 \leq r \leq n \\ i_1 + \dots + i_r = n}} m_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r}), \quad (16)$$

for any $n > 0$ and $u = n + 1 - s$.

Given a dga A , one may construct the space $B = H^*(A)$.

Thanks to a theorem by Kadeishvili, we may define an A_∞ structure on B such that

1. There is an A_∞ morphism f from B to A with f_1 equal to an embedding $i : B \hookrightarrow A$.
2. $m_1 = 0$ (i.e. B is minimal).

This A_∞ structure on B is unique up to A_∞ -isomorphism.

The combinatorics of constructing B from A is identical to a tree-level ϕ^3 quantum field theory (Kontsevich and Soibelman).

m_k is given by a sum over trees with k “in” legs and 1 “out” leg.

The vertex is given by m_2 .

The propagator H is given by “the inverse” of m_1 . More precisely, if $p : A \rightarrow B$ is a map such that $p \circ i = 1$, then

$$1 - i \circ p = m_1 H + H m_1.$$

Witten showed that the correlation functions required for our superpotential could be computed, at tree-level, by a cubic field theory:

Holomorphic Chern–Simons theory:

$$S = \int_X \text{Tr} \left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A \right) \wedge \Omega, \quad (17)$$

That is,

the A_∞ -algebra for the correlation functions is a minimal A_∞ -algebra computed from a dga given by differential forms with $m_1 = \bar{\partial}$ and $m_2 = \wedge$.

To be more precise, these differential forms are valued in $\text{End}(E)$ for a single 9-brane; or $\text{Hom}(E, F)$ for the quiver version with several 9-branes.

One may show that differential forms with $m_1 = \bar{\partial}$ and $m_2 = \wedge$ may be replaced by:

Čech cochains with $m_1 = \delta$ and $m_2 = \cup$ or

a similar structure in $\mathbf{D}(X)$ given by maps between complexes. m_1 is given by commuting this map with the maps within the complex. Thus, m_1 -cohomology corresponds to chain maps modulo homotopy. m_2 is composition of maps.

This latter structure is an intrinsic A_∞ -structure in $\mathbf{D}(X)$ (studied by Polishchuk for example). It allows us to extend the computation of the superpotential over the whole of $\mathbf{D}(X)$ (not just 9-branes).

Yet another presentation of the same mathematics is given by another, more practical picture.

Let \mathcal{E}^\bullet be a complex of *locally-free* sheaves representing a given D-brane.

Now build a double complex with entries

$$\bigoplus_{p+q=n} \check{C}^p(\mathcal{U}, \mathcal{H}om^q(\mathcal{E}^\bullet, \mathcal{E}^\bullet)), \quad (18)$$

with m_1 and m_2 from above.

This can be generalized to several branes in the obvious way.

For example, consider a 3-brane on a conifold point in X obtained by shrinking down a curve C with normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.

By Π -stability arguments, this 3-brane decays marginally into two 5-branes \mathcal{O}_C and $\mathcal{O}_C(-1)[1]$. Thus, if we considered N coincident 3-branes at this conifold point, we would have a $U(N) \times U(N)$ quiver gauge theory.

To produce a local model for this case, let X be the total space of the normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Thus we have bundle map $\pi : X \rightarrow C$. An affine open cover of X is then given by two patches: U_0 , with coordinates (x, y_1, y_2) ; and U_1 , with coordinates (w, z_1, z_2) . The transition functions are obviously

$$\begin{aligned}w &= x^{-1} \\z_1 &= xy_1 \\z_2 &= xy_2\end{aligned}\tag{19}$$

Now \mathcal{O}_C is not a locally-free sheaf on X . Define $\mathcal{O}(1) = \pi^* \mathcal{O}_C(1)$. We then have an exact sequence

$$\mathcal{O}(2) \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \\ -z_2 \\ z_1 \end{pmatrix}} \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{\begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}} \mathcal{O} \longrightarrow \mathcal{O}_C, \quad (20)$$

where we have given the explicit sheaf maps in both patches. This provides the locally-free resolution of \mathcal{O}_C , and thus $\mathcal{O}_C(-1)[1]$ too by tensoring the resolution by $\mathcal{O}(-1)$ and shifting one place to the left.

The classes in $\text{Ext}^1(\mathcal{O}_C(-1)[1], \mathcal{O}_C)$ are represented by elements of $\check{C}^0(\mathcal{U}, \mathcal{H}om^1(\mathcal{O}_C(-1)[1], \mathcal{O}_C))$ as follows. Using the notation described above, let one generator of this group, denoted a , be represented by

$$\begin{array}{ccccc}
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1) \\
 \downarrow 1 & & \downarrow -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \downarrow 1 \\
 \mathcal{O}(2) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O}(1) \oplus \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}
 \end{array} \tag{22}$$

and b by the same thing with 1 replaced by x in the vertical maps.

Next, the two generators of $\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(-1)[1])$ can be represented by elements of $\check{C}^1(\mathfrak{U}, \mathcal{H}om^0(\mathcal{O}_C, \mathcal{O}_C(-1)[1]))$. Let c be represented by

$$\begin{array}{ccccc}
 \mathcal{O}(2) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O}(1) \oplus \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O} & (23) \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \begin{matrix} 0 \\ -\frac{1}{x} & 01 \end{matrix} & & \begin{pmatrix} \frac{1}{x} & 0 \end{pmatrix}_{01} \\
 & & \downarrow & & \downarrow \\
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1)
 \end{array}$$

and d by something similar.

Finally, the generator of $\text{Ext}^3(\mathcal{O}_C(-1)[1], \mathcal{O}_C(-1)[1])$ can be represented by a 1-cochain in

$\check{C}^1(\mathcal{U}, \mathcal{H}om^2(\mathcal{O}_C(-1)[1], \mathcal{O}_C(-1)[1]))$:

$$\begin{array}{ccc}
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} \mathcal{O}(-1) \\
 & & \downarrow \begin{pmatrix} 1 \\ x \end{pmatrix}_{01} \\
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} \mathcal{O}(-1)
 \end{array}$$

(24)

The composition $c \star a$ gives a map

$$\begin{array}{ccc}
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1) & (25) \\
 \downarrow & & & & \downarrow & \\
 & & \begin{matrix} 0 \\ -\frac{1}{x} & 01 \end{matrix} & & \begin{matrix} \left(-\frac{1}{x} \ 0\right)_{01} \end{matrix} & \\
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1) & \\
 \end{array}$$

This is exact.

To be precise, $c \star a$ is a Čech coboundary of the map which is zero in patch 0 and in patch 1 given by the chain map

$$\begin{array}{ccc}
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1) & (26) \\
 \downarrow & & \downarrow & & \\
 & \begin{pmatrix} 0 \\ -1 \end{pmatrix}_1 & & \begin{pmatrix} -1 & 0 \end{pmatrix}_1 & \\
 & \downarrow & & \downarrow & \\
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1) &
 \end{array}$$

The above represents $H(c \star a)$.

Continuing this way, $b \star H(c \star a) + H(b \star c) \star a$ is

$$\begin{array}{ccccc}
 \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O} \oplus \mathcal{O} & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O}(-1) & (27) \\
 \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} -1 & 0 \end{pmatrix} & & \\
 \mathcal{O}(2) & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & \mathcal{O}(1) \oplus \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & \mathcal{O} &
 \end{array}$$

When composed with d this gives the Ext^3 of (24) but when composed with c it gives zero. Thus $m_3(b, c, a)$ is Serre dual to d .

Denoting by A the $N = 1$ superfield dual to a etc., we thus have a term in the superpotential equal to $\text{Tr}(BCAD)$.

Checking all combinations (and being careful with signs!) one obtains:

$$W = \text{Tr}(BCAD - ACBD).$$

in agreement with Klebanov and Witten.

Conclusion

- This computation can be, and has been, extended to many other examples.
- It is applicable to any set of B-type D-branes in any Calabi–Yau threefold.
- It is not clear how much stamina is required for complicated examples!
- It is only valid to tree-level.