A. 0.5 kg of a hot metal at 80°C is dropped into a large pool of water at 20°C. The specific heat of the metal at constant pressure is independent of temperature, with a value of $C_V = 100 \text{ J/(kg-K)}$.

(a) How much does the entropy of the metal change? How much does the total entropy (of both the metal and water) change? Does it increase or decrease?

(b) Calculate the ratio $\Omega_{\text{final}}/\Omega_{\text{initial}}$, where $\Omega_{\text{final}}$ and $\Omega_{\text{initial}}$ are the multiplicities of the final equilibrium macropartition and the initial non-equilibrium macropartition, respectively.

B.

(a) Estimate the temperature of the Earth assuming the Sun and Earth are blackbodies ($\alpha=1$). Take the surface temperature of the Sun to be $T_{\text{Sun}}=5800\text{ K}$, the radius of the Sun to be $R_{\text{Sun}}=7\times10^5 \text{ km}$, and the Sun-Earth distance to be $D=1.5\times10^5 \text{ km}$. Does your answer depend on the size of the Earth?

(b) Actually the Earth reflects 30% of the visible light incident upon it ($\alpha=0.7$). How does this fact alter your estimate of the Earth’s temperature?

(c) Finally, assume the Earth is surrounded by a thin spherical shell of gas that is transparent to the Sun’s energy but opaque to infrared radiation. What then is the temperature of the Earth’s surface? Note: the shell radiates energy both in space and back toward the surface of the Earth (see the Figure).
C1. The aim of this problem is to obtain the blackbody radiation relation, $E(T,V) \propto VT^4$ starting from the equation of state, by performing an infinitesimal Carnot cycle on the photon gas.

![Diagram of Carnot cycle]

(a) Express the work done, $W$, in the above cycle, in terms of $dV$ and $dP$.

(b) Express the heat absorbed, $Q$, in expanding the gas along an isotherm, in terms of $P$, $dV$, and an appropriate derivative of $E(T,V)$.

(c) Using the efficiency of the Carnot cycle, relate the above expressions for $W$ and $Q$ to $T$ and $dT$.

(d) Observations indicate that the pressure of the photon gas is given by $P = A T^4$, where $A = \pi^2 k_B^4 / 45(hc)^3$ is a constant. Use this information to obtain $E(T,V)$, assuming $E(0,V) = 0$.

(e) Find the relation describing the adiabatic paths in the above cycle.

C2. Consider a non-relativistic Fermi gas in two dimensions: $N$ electrons are confined to a square with area $A = L^2$

(a) Derive the density of electronic states in two dimensions per unit area, $g^{2D}(\varepsilon) = m/\pi \hbar^2$.

(b) Find the Fermi energy (in terms of $N$ and $A$).

(c) Find the average energy per electron at $T = 0$ (in terms of $E_F$).

(d) Calculate $E_F$ for the electron density $N/A = 10^{16} \text{ m}^{-2}$ - a typical density of electrons in a field-effect transistor (FET), assume that the effective mass is 0.2 $m_0$. Is the electron gas in a FET degenerate at room temperature?

D1. By shining an intense laser beam on a semiconductor, one can create a metastable collection of electrons (charge $-e$, and effective mass $m_e$) and holes
(charge $+e$, and effective mass $m_h$) in the bulk. The oppositely charged particles may pair up (as in a hydrogen atom) to form a gas of excitons, or they may dissociate into a plasma. We shall examine a much simplified model of this process.

(a) Calculate the free energy of a gas composed of $N_e$ electrons and $N_h$ holes, at temperature $T$, treating them as classical non-interacting particles of masses $m_e$ and $m_h$.

(b) By pairing into an exciton, the electron hole pair lowers its energy by $\Delta E$. [The binding energy of a hydrogen-like exciton is $\varepsilon \approx m e^4/(2\hbar^2 e^2)$ where $\varepsilon$ is the dielectric constant, and $m^{-1} = m_e^{-1} + m_h^{-1}$.] Calculate the free energy of a gas of $N_p$ excitons, treating them as classical non-interacting particles of mass $m = m_e + m_h$.

(c) Calculate the chemical potentials $\mu_e$, $\mu_h$, and $\mu_p$ of the electron, hole, and exciton states, respectively.

(d) Express the equilibrium condition between excitons and electron/holes in terms of their chemical potentials.

(e) At a high temperature $T$, find the density $n_p$ of excitons, as a function of the total density of excitations $n \approx n_e + n_h$. 