QM – A Solution

A) According to uncertainty principle, $\Delta p \bullet \Delta x \approx \hbar$. Therefore we expect the most probable values of the momentum to lie within $\pm\hbar/L$.

B) A wave function of a particle with certain momentum should be a eigenfunction of the momentum operator, with the value of the momentum being its eigenvalue. Therefore, for momentum p_0 the wavefunction should have a form

$$\psi_p(x) = C \cdot e^{ip_0 x/\hbar}$$

The probability to measure momentum p_0 is then a square of

$$a_p = \int_{-\infty}^{\infty} \psi_p(x)\psi(x)\,dx$$

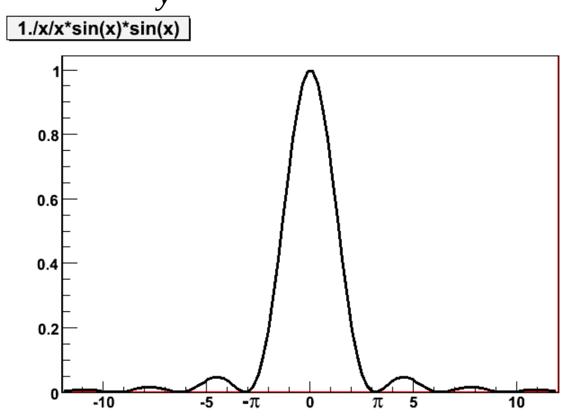
Let's calculate a_p:

$$a_p = \int_{-L}^{L} C \cdot e^{ip_0 x/\hbar} \cdot \frac{1}{\sqrt{2L}}\,dx = \frac{C}{\sqrt{2L}} \frac{\hbar}{ip_0}\left(e^{ip_0 L/\hbar} - e^{-ip_0 L/\hbar}\right) =$$

$$= C\sqrt{\frac{2}{L}} \frac{\hbar}{p_0} \sin\frac{p_0 L}{\hbar}$$

$$\wp(p_0) = a_p^2 = C^2 \frac{2\hbar^2}{Lp_0^2} \sin^2\frac{p_0 L}{\hbar}$$

Let's study the function $\dfrac{\sin^2 y}{y^2}$:



As one can see, most of the distribution is contained within $|y|<2$, so for our probability distribution we get $\left|\dfrac{p_0 L}{\hbar}\right| \le 2$, or $|p_0| \le \dfrac{2\hbar}{L}$, which is within a factor of two from the estimate in A)

C) The wavefunction right after the measurement will be ψ_p from B). To get the value of the constant C one can use the relation $\int \psi_p \psi_q dx = \delta(p-q)$, but it is easier to use the formula for probability obtained above:

$$\int \wp(p)dp = 1$$

$$\frac{2C^2\hbar^2}{L}\int_{-\infty}^{\infty}\frac{1}{p_0^2}\sin^2\frac{p_0^2 L}{\hbar}dp_0 = 2C^2\hbar\int_{-\infty}^{\infty}\frac{\sin^2 y}{y^2}dy = 1$$

Integrating by parts, the last integral can be shown to equal to

$$\int_{-\infty}^{\infty}\frac{\sin^2 y}{y^2}dy = \int_{-\infty}^{\infty}\frac{\sin y}{y}dy = \pi$$

Therefore $2\pi\hbar C^2 = 1$, and

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ip_0 x/\hbar}$$

QM - B Solution

The general form of the wave function is

$Ae^{ikx} + Be^{-ikx}$	for	$x \leq 0$
$Ce^{ik'x} + de^{-ik'x}$	for	$0 \leq x \leq a$
Fe^{ikx}	for	$x \geq a$

with $k = \sqrt{2mE}/\hbar$ and $k' = \sqrt{2m(E + |V_0|}/\hbar$

At the boundaries $x = 0$ and $x = a$, the wave function and its first derivative are continuous. Using also the fact that, in this case, there is no reflected wave, $B = 0$, we have.

$$A = C + D \tag{1}$$

$$A = \frac{k'}{k}(C - D) \tag{2}$$

$$Ce^{ik'a} + De^{-ik'a} = Fe^{ika} \tag{3}$$

$$Ce^{ik'a} - De^{-ik'a} = \frac{k}{k'}Fe^{ika} \tag{4}$$

Setting $|A|^2 = |F|^2$ we have from (1) and (3)

$$|C + D|^2 = |Ce^{ik'a} + De^{-ik'a}|^2$$

$$\Rightarrow \quad |C|^2 + |D|^2 + 2\mathrm{Re}[CD^*] = |C|^2 + |D|^2 + 2\mathrm{Re}[CD^* e^{2ik'a}]$$

Choose the overall phase so that C is real. Then (1) and (2) give

$$C + D = \frac{k'}{k}(C - D) \quad \Rightarrow \quad D \text{ is also real}$$

$$\Rightarrow \quad C^2 + D^2 + 2CD = C^2 + D^2 + 2CD\mathrm{Re}[e^{2ik'a}]$$

$$\Rightarrow \quad \cos 2k'a = 1 \quad \Rightarrow \quad k' = \frac{n\pi}{a} \quad \Rightarrow \quad E = \frac{n^2\pi^2\hbar^2}{2ma^2} - |V_0|$$

QM - C1 Solution

The simplest example of a bound state of two particles is the bound system of an electron and a positron (anti-electron) called positronium.

a) Using what you know about the hydrogen atom, determine the ground state energy of positronium and the size (average electron-positron separation) of the ground state.

> The only constants in the problem are: $\hbar$, c, the mass m, and the electron charge e.
>
> The only combination of the above constants that sets an energy scale is mc^2.
>
> The only combination of the above constants that sets a distance scale $\hbar/mc$.
>
> The only difference in the constants between the hydrogen atom and positronium is the mass. For hydrogen it is the electron mass m_e (the proton can be considered to have approximately infinite mass) while for positronium it is the reduced mass
>
> $$m = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2}$$
>
> The ground state energy of positronium is $E_0 = -6.8eV$, half of the energy of the ground state of hydrogen.
>
> The electron-positron separation in the positronium ground state is 10^{-8} cm, twice the radius of the ground state of hydrogen.

b) The ground state of positronium decays into a two-photon final state with a lifetime of about 0.1 ns. Find the expression for the polarization part of the coherent two-photon final state. You'll want to use the fact that the ground state of positronium is a state of odd (negative) parity since it is a $1S$ ($L = 0$) state and the intrinsic parity of it's two constituents, the electron and positron, are opposite.

> Since the Hamiltonian is invariant under parity transformation, the ground state is an eigenstate of parity.
>
> Since the ground state orbital angular momentum is zero, the parity is the product of the intrinsic parities of the electron and positron $\Rightarrow\ P = -1$.
>
> Since electromagnetic interactions are invariant under parity transformations, parity is conserved an d the final two photon state is also a state of negative parity.
>
> Let the line along which the photons are emitted be the z-axis. By angular momentum conservation, if the photon emitted along the $+z$ direction is right-handed polarized ($L_z = +\hbar$) then the photon emitted along the $-z$ direction is also right-harnded polarized ($L_z = -\hbar$).
>
> Under the parity transformation, the direction of momentum changes while the direction of angular momentum stays fixed. $\Rightarrow$ the two photon state $|R\,R\rangle$ transforms to the $|L\,L\rangle$.
>
> In order for the final state $|\psi\rangle$ to be a state of parity eigenstate with negative parity, we must have
> $$|R\,R\rangle \ - \ |L\,L\rangle$$

c) Joe and Sarah set up detectors on opposite sides of the positronium to measure the polarization of the emitted photons. If Joe measures the polarization of his photon to be right-handed, what is the probability that Sarah will find her photon to be x-polarized?

> $$\langle x|R\rangle \ = \ \frac{1}{\sqrt{2}}\,\langle x\big(|x\rangle + i|y\rangle\big) \ = \ \frac{1}{\sqrt{2}} \qquad\qquad \langle x|L\rangle \ = \ \frac{1}{\sqrt{2}}\,\langle x\big(|x\rangle - i|y\rangle\big) \ = \ \frac{1}{\sqrt{2}}$$
> $$|\langle R\,x|R\,R\rangle \ - \ \langle R\,x|L\,L\rangle|^2 \ = \ |\langle R|R\rangle\langle x|R\rangle \ - \ \langle R|L\rangle\langle x|L\rangle|^2 \ = \ |\langle x|R\rangle|^2 \ = \ \frac{1}{2}$$

d) If Joe measures the polarization of his photon to be x-polarized, what is the probability that Sarah will find her photon to also be x-polarized?

> $$|\langle x\,x|R\,R\rangle \ - \ \langle x\,x|L\,L\rangle|^2 \ = \ |\langle x|R\rangle\langle x|R\rangle \ - \ \langle x|L\rangle\langle x|L\rangle|^2 \ = \ \frac{1}{2} - \frac{1}{2} \ = \ 0$$

QM-C2 SOLUTION

(a) ~~$\psi(x,y,z) = \left(\frac{2}{a}\right)^{\frac{3}{2}} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}z\right)$~~

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_z^2\right) + U(x,y,z)$$

$$= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x,y,z)$$

$$H\psi(x,y,z) = E\psi(x,y,z)$$

$$\psi(x,y,z) = X(x)Y(y)Z(z)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}X(x) = E_x X(x)$$ with boundary condition.

$$X(x=0) = X(x=a) = 0.$$

$$\Rightarrow \frac{d^2}{dx^2}X(x) + k^2 X(x) = 0, \quad E_x = \frac{\hbar^2}{2m}k_x^2$$

$$X(x) = \sqrt{\frac{2}{a}}\sin kx, \quad k = \frac{n_x\pi}{a}$$

$$\Rightarrow E_x = \frac{\hbar^2}{2m}\left(\frac{n_x\pi}{a}\right)^2$$

Similary for $Y(y)$ and $Z(z)$

So
$$E = E_x + E_y + E_z$$
$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$\psi(x,y,z) = \left(\frac{2}{a}\right)^{\frac{3}{2}}\sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right)$$

$$, \quad n_x, n_y, n_z = 1, 2, 3, \ldots$$

(b) Ground state is $(n_x, n_y, n_z) = (1, 1, 1)$ ②

$$\Rightarrow \psi_{111} = \left(\frac{2}{a}\right)^{\frac{3}{2}} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{a}y\right) \sin\left(\frac{\pi}{a}z\right).$$

$$E_{111} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \cdot 3.$$

First order correction for the ground state energy is

$$E^1_{111} = \langle \psi_{111} | V' | \psi_{111} \rangle$$

$$= a^3 V_0 \left(\frac{2}{a}\right)^3 \int_0^a dx \int_0^a dy \int_0^a dz \, \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{a}y\right) \sin^2\left(\frac{\pi}{a}z\right) \times \delta\left(x - \frac{a}{4}\right) \delta\left(y - \frac{a}{2}\right) \delta\left(z - \frac{3a}{4}\right)$$

$$= a^3 V_0 \left(\frac{2}{a}\right)^3 \Bigg[\underbrace{\sin^2\left(\frac{\pi}{4}\right)}_{= \frac{1}{2}} \underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{= 1} \underbrace{\sin^2\left(\frac{3\pi}{4}\right)}_{= \frac{1}{2}} \Bigg].$$

$$= 8 V_0 \cdot \frac{1}{4} = 2 V_0$$

The first excited states are

$(2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$

Let's define

$|\psi_1\rangle \equiv |\psi_{211}\rangle$

$|\psi_2\rangle = |\psi_{121}\rangle$, $|\psi_3\rangle = |\psi_{112}\rangle$.

$$\langle \psi_1 | V' | \psi_1 \rangle = 8V_0 \cdot \Big[\underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1} \underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1} \underbrace{\sin^2\left(\frac{3\pi}{4}\right)}_{\frac{1}{2}} \Big]$$

$$= 4V_0$$

$$\langle \psi_2 | V' | \psi_2 \rangle = 8V_0 \Big[\sin^2\left(\frac{\pi}{4}\right) \sin^2(\pi) \sin^2\left(\frac{3\pi}{4}\right) \Big)$$

$$= 0$$

$$\langle \psi_3 | V' | \psi_3 \rangle = 8V_0 \Big[\underbrace{\sin^2\left(\frac{\pi}{4}\right)}_{\frac{1}{2}} \underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1} \underbrace{\sin^2\left(\frac{3\pi}{2}\right)}_{1} \Big]$$

$$= 4V_0$$

$$\langle \psi_1 | V' | \psi_2 \rangle = 8V_0 \Big[\sin\left(\frac{2\pi}{a}\frac{a}{4}\right) \sin\left(\frac{\pi}{a}\frac{a}{4}\right) \cdot \sin\left(\frac{\pi}{a}\frac{a}{2}\right) \sin\left(\frac{2\pi}{a}\frac{a}{2}\right) \cdot \sin^2\left(\frac{3\pi}{4}\right) \Big]$$

$$= 8V_0 \Big[\frac{1}{\sqrt{2}} \cdot 0 \; \frac{1}{2} \Big] = 0$$

$$\langle \psi_1 | V' | \psi_3 \rangle = 8V_0 \Big[\sin\left(\frac{2\pi}{a}\frac{a}{4}\right) \sin\left(\frac{\pi}{a}\frac{a}{4}\right) \sin^2\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{a}\frac{3a}{4}\right) \sin\left(\frac{2\pi}{a}\cdot\frac{3a}{4}\right) \Big]$$

$$= 8V_0 \Big[\frac{1}{\sqrt{2}} \cdot 1 \cdot \left(-\frac{1}{\sqrt{2}}\right)$$

$$= -4V_0$$

$$\langle \psi_2 | V' | \psi_3 \rangle = 8V_0 \Big[\sin^2\left(\frac{\pi}{2}\right) \overset{0}{\sin\left(\frac{2\pi}{\cancel{a}}\frac{\cancel{a}}{2}\right)} \cdots \Big]$$

$$= 0$$

So the perturbation Hamiltonian matrix is

$$H' = \begin{pmatrix} 4V_0 & 0 & -4V_0 \\ 0 & 0 & 0 \\ -4V_0 & 0 & 4V_0 \end{pmatrix} = 4V_0 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

$$H'|\psi\rangle = 4V_0 \lambda |\psi\rangle .$$

$\Rightarrow$ characteristic equation is

$$\det \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & -\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\big(\lambda(\lambda-1)\big) - (-\lambda) = 0$$

$$\Rightarrow (\lambda-1)^2\lambda - \lambda = 0 \Rightarrow \lambda(\lambda-1-1)(\lambda-1+1) = 0$$

$$\Rightarrow \lambda^2(\lambda-2) = 0$$

$\Rightarrow \lambda = 0$ (doubly degenerate) or 2.

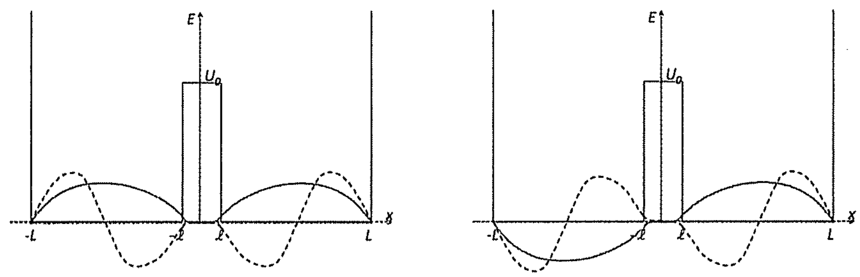
So the first order corrections to the first excited energy are $\underline{0,\ 0,\ 8V_0}$.

QM – D1 Solution

A) From general properties of Schroedinger's equation:

- Since the potential is an even function of x, the wavefunctions will be either even or odd functions of x.
- For l<|x|<L the solution is a sine wave and for the |x|<l the wavefunction is exponentially decreasing.
- The boundary conditions are that ψ and ψ' are contiguous functions between $-L$ and L, and $\psi(L)=\psi(-L)=0$.

Therefore the four lowest levels (two even, two odd) will be:



B) Let us use A) to write down general representations of the wavefunctions:

$$even:\quad \begin{cases} \psi_{l+L}(x) = a\cdot \sin k(L-x) \\ \psi_{-l+l}(x) = b\cdot (e^{\alpha x}+e^{-\alpha x}) \\ \psi_{-L+-l}(x) = a\cdot \sin k(L+x) \end{cases}$$

$$odd:\quad \begin{cases} \psi_{l+L}(x) = a\cdot \sin k(L-x) \\ \psi_{-l+l}(x) = b\cdot (e^{\alpha x}-e^{-\alpha x}) \\ \psi_{-L+-l}(x) = -a\cdot \sin k(L+x) \end{cases}$$

where $k=\dfrac{\sqrt{2mE}}{h}$ and $\alpha=\dfrac{\sqrt{2m(U_0-E)}}{h}\approx\dfrac{\sqrt{2mU_0}}{h}$. The coefficients a and b are determined by the boundary conditions and the total normalization of the wavefunctions. Let us consider the requirement of continuity of ψ and ψ':

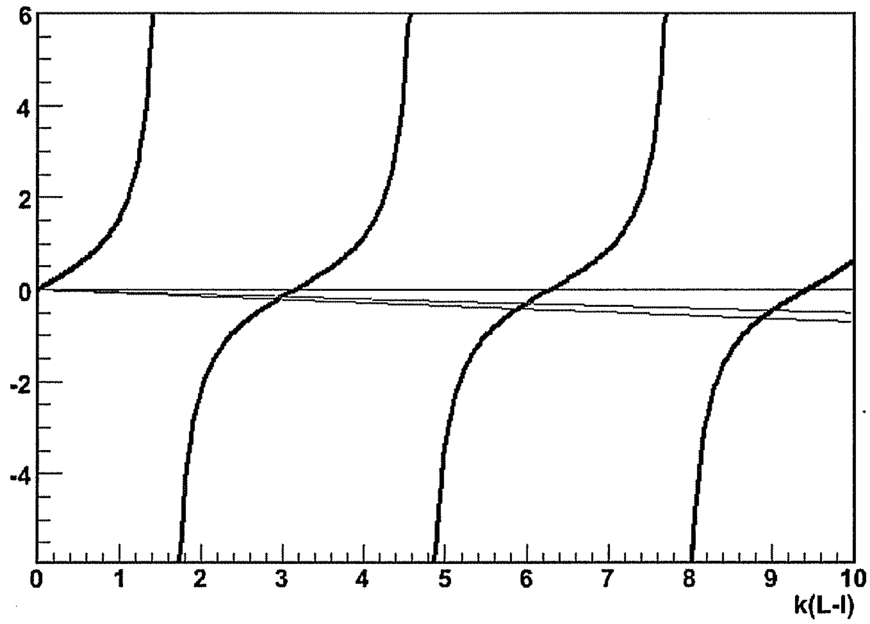
$$even:\quad \begin{cases} a\cdot \sin k(L-l) = b\cdot (e^{\alpha x}+e^{-\alpha x}) \\ -k\cdot a\cdot \cos k(L-l) = b\cdot \alpha\cdot (e^{\alpha x}-e^{-\alpha x}) \end{cases}$$

$$odd:\quad \begin{cases} a\cdot \sin k(L-l) = b\cdot (e^{\alpha x}-e^{-\alpha x}) \\ -k\cdot a\cdot \cos k(L-l) = b\cdot \alpha\cdot (e^{\alpha x}+e^{-\alpha x}) \end{cases}$$

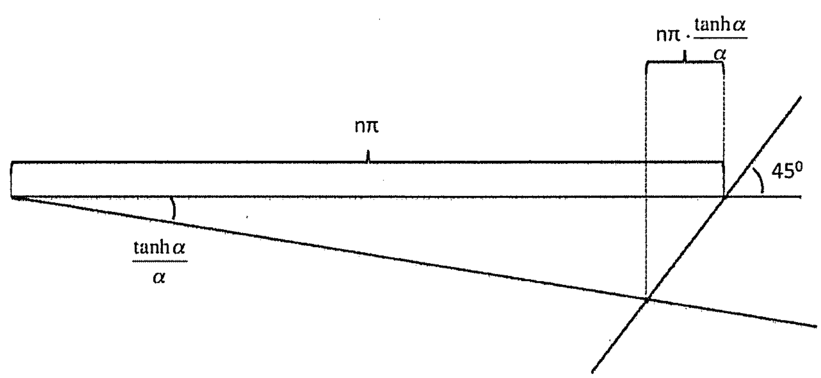
Dividing on by the other we get

$$\tan k(L-l) = -k\frac{1}{\alpha \tanh \alpha l} \qquad \text{and} \qquad \tan k(L-l) = -k\frac{\tanh \alpha l}{\alpha}$$

Since α is large, this equation can be approximately solved by approximating tan $k(L\text{-}l)$ around $2\pi n$ as shown in the figure below:



The solutions will be



$$k = n\frac{\pi}{L-l}\left(1 - \frac{1}{(L-l)\cdot\alpha\cdot\tanh \alpha l}\right) \qquad (even)$$

$$, \text{ and } E = \frac{\hbar^2}{2m}k^2$$

$$k = n\frac{\pi}{L-l}\left(1 - \frac{\tanh \alpha l}{(L-l)\cdot\alpha}\right) \qquad (odd)$$

C) Even and odd level's energies are close to each other and close to the energy levels of an infinitely deep rectangular well with width $(L\text{-}l)$. When

U_0 is infinitely large, the problem becomes a system of two infinitely deep wells. The energy levels are degenerate, and correspond to the particle motion either in the left or in the right well. When U_0 is finite, so is the probability for a particle to tunnel from one side to the other. The stationary states then start corresponding to the particle moving in both wells, and the energy levels split.

QM-D2 SOLUTION

The perturbation Hamiltonian is

$$H'(t) = -q\vec{E}(t)\cdot\vec{r} = eE_0 e^{-\gamma t} z \quad \text{for } t>0$$

if we assume $\vec{E}_0 = E_0\hat{z}$

Using the 1st-order time-dependent perturbation theory, the transition amplitude from an initial state to a final state $\dot{c}_{i\to f}$ is given by

$$i\hbar \frac{dC_{i\to f}}{dt} = \langle f|H'(t)|i\rangle e^{i(E_f - E_i)t/\hbar}$$

Here the initial state is $|1s\rangle \equiv |100\rangle$ and final state is one of the three $|2p\rangle$ states, that is $|210\rangle$ and $|21\pm1\rangle$.

Since $\langle f|H'(t)|i\rangle = eE_0 e^{-\gamma t}\langle f|z|i\rangle$, we need to consider the three matrix elements, $\langle 210|z|100\rangle$ and $\langle 21\pm1|z|100\rangle$.

②

With the spherical coordinates, $z = r\cos\theta$,

and $\langle 21\,{}^{0}_{\pm1} | z | 100\rangle = \langle R_{21} Y_1^{0,\pm1} | r\cos\theta | R_{10} Y_0^0\rangle$

$= \langle R_{21} | r | R_{10}\rangle \langle Y_1^{0,\pm1} | \cos\theta | Y_0^0\rangle$.

Here ~~lets~~ let's first consider the angular integration $\langle Y_1^{0,\pm1} | \cos\theta | Y_0^0\rangle$.

With $Y_0^0 = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$, ~~and~~ $Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$

and $Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{\pm i\phi}$,

$$\langle Y_1^0 | \cos\theta | Y_0^0\rangle = \frac{\sqrt{3}}{4\pi} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \cos^2\theta \cdot \sin\theta \, d\theta\, d\phi$$

$$\cancel{= \frac{\sqrt{3}}{4} \int_{\theta=0}^{\pi} \sin 2\theta \, d\theta = 0}$$

$$= \frac{\sqrt{3}}{2} \int_{\theta=0}^{\pi} \cos^2\theta \, (-d\cos\theta)$$

$$= \frac{\sqrt{3}}{2} \left(-\frac{\cos^3\theta}{3}\right)\Bigg|_{\theta=0}^{\pi} = \frac{1}{\sqrt{3}}$$

$$\langle Y_1^{\pm1} | \cos\theta | Y_0^0\rangle = \frac{1}{\mp 4\pi}\sqrt{\frac{3}{2}} \int_\theta\int_\phi \sin\theta\, e^{\mp i\phi} \cdot \cos\theta \sin\theta \, d\theta\, d\phi$$

$$= \mp\frac{1}{4\pi}\sqrt{\frac{3}{2}} \int_\theta \sin^2\theta \cos\theta \, d\theta \int_{\phi=0}^{2\pi} e^{\mp i\phi} d\phi$$

(3)

Because $\int_{\phi=0}^{2\pi} e^{\mp i\phi} d\phi = 0$,

$\langle Y_1^{\pm 1} | \cos\theta | Y_0^0 \rangle = 0$.

Therefore, the only nonvanishing matrix element is $\langle Y_1^0 | \cos\theta | Y_0^0 \rangle = \frac{1}{\sqrt{3}}$.

Now $\langle R_{21} | r | R_{10} \rangle = \int_{r=0}^{\infty} R_{21}^* R_{10} r^3 dr$

$$= \int_{r=0}^{\infty} 2a^{-3} \exp\left(-\frac{r}{a}\right) \exp\left(-\frac{r}{2a}\right) \cdot \frac{r}{a} \frac{1}{\sqrt{24}} r^3 dr$$

$$= \frac{1}{\sqrt{6}} \cdot \frac{1}{a^4} \int_{r=0}^{\infty} \exp\left(-\frac{3r}{2a}\right) r^4 dr$$

$$= \frac{1}{\sqrt{6}} \frac{1}{a^4} \cdot 4! \left(\frac{2a}{3}\right)^5 = \cancel{\frac{4!}{\sqrt{6}}} \frac{a}{\sqrt{6}} \frac{2^8}{3^4} \cancel{\frac{1}{a^4}}$$

Thus $\langle 210 | z | 100 \rangle$

$$= \langle R_{21} | r | R_{10} \rangle \langle Y_1^0 | \cos\theta | Y_0^0 \rangle$$

$$= \frac{a}{\sqrt{6}} \frac{2^8}{3^4} \cdot \frac{1}{\sqrt{3}} = \frac{a}{\sqrt{2}} \frac{2^8}{3^5}$$

Now

$$i\hbar \frac{dC_{i\to f}}{dt} = \langle f|H'(t)|i\rangle e^{i\frac{(E_f-E_i)t}{\hbar}}$$

$$= eE_0 e^{-\gamma t}\langle f|z|i\rangle e^{i\omega_0 t}$$

$$\Rightarrow C_{i\to f}(t) = eE_0\langle f|z|i\rangle\cdot\left(\frac{1}{i\hbar}\right)\int_0^t e^{(i\omega_0-\gamma)t'}dt'$$

$$= -i\frac{eE_0\langle f|z|i\rangle}{\hbar}\cdot\frac{e^{(i\omega_0-\gamma)t}-1}{i\omega_0-\gamma}.$$

For $t\to\infty$

$$C_{i\to f}(t=\infty) = +i\frac{eE_0\langle f|z|i\rangle}{\hbar}\cdot\frac{1}{i\omega_0-\gamma}.$$

So

$$P_{i\to f}(t=\infty) = |C_{i\to f}(t=\infty)|^2 \cancel{= \frac{eE_0\langle f}{\hbar}}$$

$$= \frac{e^2E_0^2}{\hbar^2(\omega_0^2+\gamma^2)}|\langle f|z|i\rangle|^2.$$

with $i=|100\rangle$ and $f=|21,{}^{0}_{\pm1}\rangle$,

~~as we have shown above,~~

$$P_{1s\to2p} = \frac{e^2E_0^2}{\hbar^2(\omega_0^2+\gamma^2)}\Big[|\langle210|z|100\rangle|^2 + |\langle21{+1}|z|100\rangle|^2 + |\langle21{-1}|z|100\rangle|^2\Big]$$

$$= \frac{e^2 E_0^2}{\hbar^2(\omega_0^2+\gamma^2)} \cdot \frac{a^2}{2} \frac{2^{16}}{3^{10}}$$

$$= \frac{2^{15} e^2 E_0^2 a^2}{3^{10} \hbar^2(\omega_0^2+\gamma^2)}$$