CM-A Solution

(a) To the right.
(b) $T\cos\theta = F_f$. Now, $\tau = I\,\alpha \Rightarrow Tr_1 - F_f r_2 = 0$ since $\alpha = 0 \Rightarrow Tr_1 = F_f r_2$
Thus, $Tr_1 = T(\cos\theta)r_2$. $\Rightarrow \theta = \arccos(r_1/r_2)$
(c) For the critical angle and maximum tension, $F_f = \mu_s F_N$ and $F_N = mg - T_{max}\sin\theta_c$. In addition, $T_{max}\cos\theta_c = F_f = \mu_s F_N$. These lead to $F_N = mg - T_{max}\sin\theta_c$ $= T_{max}(\cos\theta_c)/\mu_s = T_{max} r_1/\mu_s r_2$. Thus, $T_{max}(r_1/\mu_s r_2 + \sin\theta_c) = mg$. $\Rightarrow$ $T_{max} = mg/[r_1/\mu_s r_2 + (r_2^2 - r_1^2)^{1/2}/r_2]$.

CM - B Solution:

Consider the rocket and exhaust just before and after the ejection of an amount of mass dm_e. The momentum of the system must be the same before and after the ejection. If the rocket has velocity v and mass m before ejection, then its momentum then is mv. After ejection, the momentum of the rocket plus exhaust is, to first order in small quantities

$$(m - dm_e)(v + dv) + dm_e(v - u_e) = mv + m\,dv - dm_e\,v - dm_e\,dv + dm_e\,v - dm_e\,u_e \quad (1)$$

$$= mv + m\,dv - dm_e\,u_e. \quad (2)$$

Equating the two momenta yields

$$mv = mv + m\,dv - dm_e\,u_e \quad (3)$$

$$\Rightarrow m\,dv = dm_e\,u_e \quad (4)$$

Now the change in the mass of the rocket is $dm = -dm_e$, so

$$\Rightarrow \int_0^v dv = -\int_{m_i}^{m_f} (dm/m)u_e \quad (5)$$

$$\Rightarrow v = \ln(m_i/m_f)u_e. \quad (6)$$

CM - C1 Solution

We consider more general case with $\beta \neq \frac{\pi}{2} - \alpha$, as shown in Figs.1, 2. Let $\vec{w}$ be the acceleration of M and $\vec{a}$ be the linear acceleration of the ball with respect to the incline. Let also $\vec{F}$, $\vec{N}$ and $\vec{F}_I = -m\,\vec{w}$ be, respectively, the friction force, the normal force and the inertial force acting on the ball in the non-inertial reference frame related to the incline. The second Newton law for the ball in the non-inertial reference frame gives (see Fig.1)

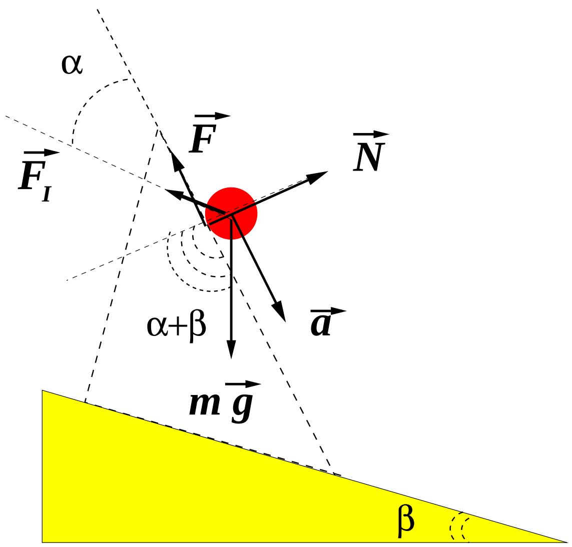


Fig. 1

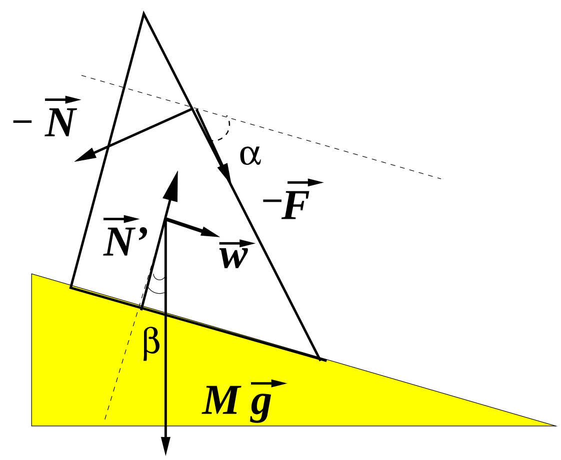


Fig. 2

$$m\,a = mg\;\sin(\alpha+\beta) - F - m\,w\;\cos(\alpha)\;, \qquad m\;g\;\cos(\alpha+\beta) + m\,w\;\sin(\alpha) = N\;.$$

An angular acceleration of the ball is given by

$$\alpha = \frac{a}{R} \qquad \text{and} \qquad I\;\alpha = F\,R\;,$$

where I is the moment of inertia of the ball. Hence

$$F = \gamma\,ma = \frac{\gamma\,m}{1+\gamma}\left[g\sin(\alpha+\beta) - w\;\cos(\alpha)\right]\;, \qquad N = m\,g\;\cos(\alpha+\beta) + m\,w\;\sin(\alpha)\;,$$

where $\gamma = \frac{I}{mR^2}$.

The second Newton law for M gives (see Fig.2):

$$\begin{aligned} M\,w &= M\,g\;\sin(\beta) + F\;\cos(\alpha) - N\;\sin(\alpha) \\ &= M\,g\;\sin(\beta) + \frac{\gamma\,m}{1+\gamma}\left[g\sin(\alpha+\beta) - w\;\cos(\alpha)\right] \\ &\quad -m\left[g\;\cos(\alpha+\beta) + w\;\sin(\alpha)\right]\;\sin(\alpha)\;. \end{aligned}$$

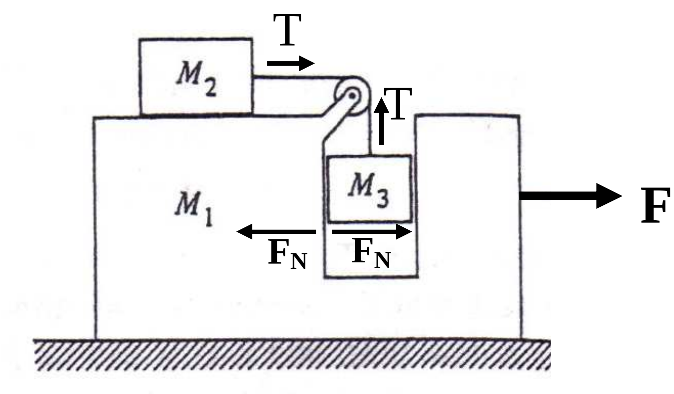
Solving this equation with respect to w, one finds

$$
\begin{aligned}
w &= g \left[\frac{\sin(\beta) + \frac{\gamma\mu}{1+\gamma} \sin(\alpha+\beta)\cos(\alpha) - \mu \sin(\alpha)\cos(\alpha+\beta)}{1 + \frac{\gamma\mu}{1+\gamma} \cos^2(\alpha) + \mu \sin^2(\alpha)} \right] \\
&= g \left[\frac{\big(1+\gamma(1+\mu)\big) \sin(\beta) - \mu \sin(\alpha) \cos(\alpha+\beta)}{(1+\gamma)(1+\mu) - \mu \cos^2(\alpha)} \right], \quad \text{with} \quad \mu = \frac{m}{M} \; .
\end{aligned}
$$

Notice that for the uniform ball $\gamma = \frac{I}{mR^2} = \frac{2}{5}$. Finally, for $\alpha + \beta = \frac{\pi}{2}$ one obtains,

$$
w = \frac{g \; (7M \; + 2\, m) \; \cos(\alpha)}{7\,(m+M) - 5\, m \; \cos^2(\alpha)} \; .
$$

CM-C2 Solution:



(a) $T=M_3g$ and $T=M_2a$, leading to $M_3g=M_2a$. Thus, $a=M_3g/M_2$

(b) $F_N=M_3a=M_3^2g/M_2$. In addition, $F-T-F_N=M_1a$.
Thus, $F=M_1a+T+F_N=M_1M_3g/M_2+M_3g+M_3^2g/M_2=(M_1+M_2+M_3)M_3g/M_2$

CM - D1 Solution

a) Adopt a coordinate system centered on the top end of the string. Let (x_{cm}, y_{cm}) be the coordinates of the center of mass of the bar, which is located half-way down its length. Let (x_1, y_1) be the coordinates of the connection between the string and bar. Then

$$\begin{aligned} x_{cm} &= x_1 + \frac{3L}{4}\sin(\theta) = L\sin(\phi) + \frac{3L}{4}\sin(\theta) && (1)\\ y_{cm} &= y_1 - \frac{3L}{4}\cos(\theta) = -L\cos(\phi) - \frac{3L}{4}\cos(\theta). && (2)\end{aligned}$$

Taking the time derivative yields the velocity of the center of mass:

$$\begin{aligned} \dot{x}_{cm} &= L\cos(\phi)\dot{\phi} + \frac{3L}{4}\cos(\theta)\dot{\theta} && (3)\\ \dot{y}_{cm} &= L\sin(\phi)\dot{\phi} + \frac{3L}{4}\sin(\theta)\dot{\theta}. && (4)\end{aligned}$$

The Lagrangian of the system is the kinetic energy of the bar minus its potential energy, $L = K - U$. The potential energy is simply related to the height of the center of mass of the bar, $U = Mgy_{cm}$. The kinetic energy is the sum of the kinetic energy of the center-of-mass motion and that of the rotational motion:

$$\begin{aligned} K &= K_{cm} + K_{rot} && (5)\\ &= \frac{1}{2}M\left((\dot{x}_{cm})^2 + (\dot{y}_{cm})^2\right) + \frac{1}{2}I(\dot{\theta})^2 && (6)\\ &= \frac{1}{2}M\left(\left(L\cos(\phi)\dot{\phi} + \frac{3L}{4}\cos(\theta)\dot{\theta}\right)^2 + \left(L\sin(\phi)\dot{\phi} + \frac{3L}{4}\sin(\theta)\dot{\theta}\right)^2\right) + \\ &\quad \frac{1}{2}\left(\frac{1}{12}M(3L/2)^2\right)(\dot{\theta})^2 && (7)\\ &= \frac{1}{2}ML^2\left[(\dot{\phi})^2 + \frac{9}{16}(\dot{\theta})^2 + \frac{3}{2}\left(\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)\right)\dot{\phi}\dot{\theta} + \frac{3}{16}(\dot{\theta})^2\right] && (8)\\ &= \frac{1}{2}ML^2\left[(\dot{\phi})^2 + \frac{3}{4}(\dot{\theta})^2 + \frac{3}{2}\cos(\phi-\theta)\dot{\phi}\dot{\theta}\right]. && (9)\end{aligned}$$

In equation 6, I is the moment of inertia of the bar. Thus, the Lagrangian is

$$L = \frac{1}{2}ML^2\left[(\dot{\phi})^2 + \frac{3}{4}(\dot{\theta})^2 + \frac{3}{2}\cos(\phi-\theta)\dot{\phi}\dot{\theta}\right] + MLg\left(\cos(\phi) + \frac{3}{4}\cos(\theta)\right) \tag{10}$$

To derive the Lagrangian for small oscillations, keep terms up to second order in the angles:

$$L = \frac{1}{2}ML^2\left[(\dot{\phi})^2 + \frac{3}{4}(\dot{\theta})^2 + \frac{3}{2}\dot{\phi}\dot{\theta}\right] + MLg\left((1-\frac{1}{2}\phi^2) + \frac{3}{4}(1-\frac{1}{2}\theta^2)\right). \quad (11)$$

b) The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \quad (12)$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0. \quad (13)$$

Substituting in the Lagrangian appropriate for small oscillations yields (using the full Lagrangian and then finding the appropriate limit yields the same result):

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}ML^2(\frac{3}{2}\dot{\theta} + \frac{3}{2}\dot{\phi})\right) - MLg\left(-\frac{3}{4}\theta\right) &= 0 & (14)\\ \Rightarrow \frac{3}{4}ML^2\left(\ddot{\theta} + \ddot{\phi}\right) + \frac{3}{4}MLg\theta &= 0 & (15)\\ \Rightarrow L\ddot{\theta} + L\ddot{\phi} + g\theta = 0. && (16) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}ML^2(2\dot{\phi} + \frac{3}{2}\dot{\theta})\right) - MLg\left(-\phi\right) &= 0 & (17)\\ \Rightarrow ML^2\left(\ddot{\phi} + \frac{3}{4}\ddot{\theta}\right) + MLg\phi &= 0 & (18)\\ \Rightarrow L\ddot{\phi} + \frac{3}{4}L\ddot{\theta} + g\phi = 0 && (19) \end{aligned}$$

c) In the normal modes, both ϕ and θ vary sinusoidally with the same frequency. Thus, the modes have the form $\phi = A_\phi \sin(\omega t)$ and $\theta = A_\theta \sin(\omega t)$. Intuitively, we expect that one mode with have the same sign for A_ϕ and A_θ and the other will have opposite signs. Plugging these forms into the equations of motion produces

$$\begin{aligned} -A_\theta L\omega^2 \sin(\omega t) - A_\phi L\omega^2 \sin(\omega t) + gA_\theta \sin(\omega t) &= 0 \\ \Rightarrow A_\theta + A_\phi = \frac{g}{L\omega^2}A_\theta \Rightarrow A_\phi &= \left(\frac{g}{L\omega^2} - 1\right)A_\theta \quad (20) \end{aligned}$$

and

$$
\begin{aligned}
-A_\phi L\omega^2 \sin(\omega t) - \frac{3}{4} A_\theta L\omega^2 \sin(\omega t) + g A_\phi \sin(\omega t) &= 0 \\
\Rightarrow A_\phi + \frac{3}{4} A_\theta = \frac{g}{L\omega^2} A_\phi \Rightarrow A_\phi &= \frac{3}{4}\left(\frac{g}{L\omega^2} - 1\right)^{-1} A_\theta \qquad (21)
\end{aligned}
$$

Equating equations 20 and 21 yields ω:

$$
\left(\frac{g}{L\omega^2} - 1\right) A_\theta = \frac{3}{4}\left(\frac{g}{L\omega^2} - 1\right)^{-1} A_\theta \Rightarrow \left(\frac{g}{L\omega^2} - 1\right)^2 = \frac{3}{4} \Rightarrow \frac{g}{L\omega^2} - 1 = \pm\sqrt{\frac{3}{4}}
$$

$$
\Rightarrow \omega = \pm \frac{\sqrt{g/L}}{\sqrt{1 \pm \sqrt{3/4}}}. \tag{22}
$$

The leading $\pm$ determines only whether the pendulum initially swings left or right. Plugging in the frequency $\omega_\pm = \sqrt{(g/L)/(1 \pm \sqrt{3/4})}$ into equation 20 yields

$$
A_\phi = (1 \pm \sqrt{\frac{3}{4}} - 1) A_\theta = \pm\sqrt{\frac{3}{4}} A_\theta. \tag{23}
$$

Thus, the two normal modes are

$$
\theta = A \sin(\omega_+ t) \qquad \phi = \sqrt{\frac{3}{4}} A \sin(\omega_+ t) \qquad \omega_+ = \frac{\sqrt{g/L}}{\sqrt{1 + \sqrt{3/4}}} \tag{24}
$$

and

$$
\theta = A \sin(\omega_- t) \qquad \phi = -\sqrt{\frac{3}{4}} A \sin(\omega_- t) \qquad \omega_- = \frac{\sqrt{g/L}}{\sqrt{1 - \sqrt{3/4}}}. \tag{25}
$$

In the low-frequency normal mode the top and bottom of the bar move in phase. In the high-frequency normal mode the top and the bottom of the bar move in opposite directions. For both normal modes the amplitude of the motion of ϕ is smaller than that of θ by $\sqrt{3/4}$.

d) A force F applied horizontally to the bottom of the vertically-hanging bar will cause it to both displace and rotate. The string will initially exert no horizontal force and, hence, no torque on the bar. The change in θ can be calculated from the torque exerted on the bar. The simplest way to determine the change in ϕ is to find the horizontal displacement of the point where the rod connects to the string, x_1. That displacement is a combination of the displacement of the center of mass of the rod and the rotation of the rod. For infinitesimal displacements,

$$\begin{aligned} \phi = x_1/L &= (x_{cm} - \theta(3L/4))/L \\ \Rightarrow \dot{\phi} &= \dot{x}_{cm}/L - (3/4)\dot{\theta}. \end{aligned} \tag{26}$$

After a time Δt, $M\dot{x}_{cm} = F\Delta t$ from Newton's second law. The force exerts a torque $F(3L/4)$ on the bar, so after Δt

$$I\dot{\theta} = F\left(\frac{3L}{4}\right)\Delta t \tag{27}$$

$$\Rightarrow \frac{1}{12}M\left(\frac{3L}{2}\right)^2\dot{\theta} = \left(\frac{3L}{4}\right)F\Delta t \tag{28}$$

$$\Rightarrow \dot{\theta} = 4\frac{F\Delta t}{ML}. \tag{29}$$

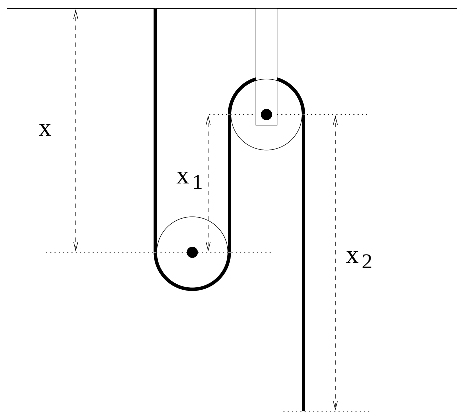
Plugging these results into equation 26 for $\dot{\phi}$ yields

$$\dot{\phi} = \frac{F\Delta t}{ML} - \frac{3}{4}\left(4\frac{F\Delta t}{ML}\right) \tag{30}$$

$$= -2\left(\frac{F\Delta t}{ML}\right). \tag{31}$$

Thus, for small Δt, $\theta = -2\phi$. Note that both normal modes are excited by the force.

CM - D2 Solution



1. Constraints:

$$x = x_1 + h\,, \qquad x + x_1 + x_2 + 2\pi R = L\,, \qquad (s.1)$$

2. Kinetic energy:

$$\text{Rope, piece } x_2: \qquad \frac{\rho\, x_2}{2}\, \dot{x}_2^2$$

$$\text{Rope, piece over } P_{II}: \qquad \frac{\rho\, \pi R}{2}\, \dot{x}_2^2$$

$$\text{Rope, piece } x_1: \qquad \frac{\rho\, x_1}{2}\, \dot{x}_1^2$$

$$\text{Rope, piece under } P_I: \qquad \rho\, \pi R\, \dot{x}^2$$

$$\text{Pulley } P_I: \qquad \frac{M}{2}\, \dot{x}^2 + \frac{I}{2R^2}\, \dot{x}^2$$

$$\text{Pulley } P_{II}: \qquad \frac{I}{2R^2}\, \dot{x}_2^2$$

where

$$I = \frac{1}{2}\, M R^2$$

is the moment of inertia of the pulley.

3. Potential energy:

$$\text{Rope, piece } x_2: \qquad -\frac{g\rho}{2}\, x_2^2$$

$$\text{Rope, piece } x_1: \qquad -\frac{g\rho}{2}\, x_1^2$$

$$\text{Rope, piece } x: \qquad -\frac{g\rho}{2}\, x^2$$

$$\text{Pulley } P_I: \qquad -gM\,x$$

4. Excluding x_1 and x_2 and the associated velocities through (s.1), one finds for the full kinetic and potential energies

$$K = \frac{1}{2}\,(A\,x + B)\,\dot{x}^2\,,$$

with

$$A = -9\,\rho\,, \qquad B = \rho(4L - 2\pi\,R + 5\,h) + M + 5I/R$$

and

$$U = -gM\,x - \frac{g\rho}{2}\,((L - 2\pi R + h - 2x)^2 + (h - x)^2 + x^2)\,.$$

5. The equation of motion then reads

$$(A\,x + B)\,\ddot{x} + \frac{A}{2}\,\dot{x}^2 + g\rho\,(2L - 4\pi R + 3h - 6x) - gM = 0$$

When the rope is released we have $\dot{x} = 0$ and $x = x_0$. Then

$$\ddot{x} = \frac{gM - g\rho(2L + 3h - 4\pi R - 6x_0)}{7M/2 + \rho(4L + 5h - 2\pi R - 9x_0)}$$