QM - A SOLUTION

a) The possible measured values for J_Z for a particle in an $j = 1$ state are $J_Z = +\hbar,\ 0,\ -\hbar$. Since the particle is in an J_Z eigenstate with eigenvalue $m\hbar = +\hbar$, the probability of measuring $J_Z = +\hbar$ is 100% with the probabilities of measuring $J_Z = 0$ or $J_Z = -\hbar$ equal to zero.

b)

$$\langle J^2 \rangle = j(j+1)\hbar^2 = 2\hbar^2 \qquad \Rightarrow \qquad \langle J_X^2 \rangle + \langle L_Y^2 \rangle + \langle L_Z^2 \rangle = 2\hbar^2$$

$$\langle J_Z^2 \rangle = m^2\hbar^2 = \hbar^2 \qquad \text{and by symmetry} \qquad \langle J_X^2 \rangle = \langle J_Y^2 \rangle$$

$$\Rightarrow \qquad \langle J_X^2 \rangle = \langle J_Y^2 \rangle = \frac{\hbar^2}{2}$$

$$\langle J_X^2 \rangle = \hbar^2 P_{\text{rob}}(J_X = +\hbar) \ + \ 0 P_{\text{rob}}(J_X = 0) \ + \ \hbar^2 P_{\text{rob}}(J_X = -\hbar)$$

$$\text{By symmetry} \qquad P_{\text{rob}}(J_X = +\hbar) = P_{\text{rob}}(J_X = -\hbar)$$

$$\Rightarrow \qquad P_{\text{rob}}(J_X = +\hbar) = P_{\text{rob}}(J_X = -\hbar) = \frac{1}{4}$$

$$\text{Since} \qquad P_{\text{rob}}(J_X = +\hbar) \ + \ P_{\text{rob}}(J_X = 0) \ + \ P_{\text{rob}}(J_X = -\hbar) = 1$$

$$P_{\text{rob}}(J_X = 0) = \frac{1}{2}$$

QM – B SOLUTION

If the girl constrains the marbles to have position very close to the desired, they will have a substantial spread in their horizontal momenta, according to the uncertainty principle:

$$\Delta p_x \geq \frac{h}{2\Delta x_0},$$

where Δx_0 is the uncertainty in initial position. This means that as the marbles fall, the uncertainty in their position is going to grow:

$$\Delta x \geq \Delta x_0 + t\frac{\Delta p_x}{M} = \Delta x_0 + t\frac{h}{2M\Delta x_0}.$$

If, on the other hand, she constraints the horizontal momentum of the particles, the initial Δx_0 is going to be large. This means that there is some optimal value of Δx_0 that minimized the final Δx.

The girl needs to minimize Δx after the time $t = \sqrt{2H/g}$ which is how long it takes for the marbles to fall. At the optimal Δx_0 the derivative d(Δx)/d(Δx_0) should be zero:

$$\frac{d(\Delta x)}{d(\Delta x_0)} = 1 - \frac{h}{2M(\Delta x_0)^2}t = 1 - \frac{h}{2M(\Delta x_0)^2}\sqrt{\frac{2H}{g}}$$

$$\Delta x_0 = \sqrt{\frac{h}{2M}}\sqrt[4]{\frac{2H}{g}}.$$

Substituting this back to the formula for Δx we get

$$\Delta x \geq \Delta x_0 + \sqrt{\frac{2H}{g}}\frac{h}{2M\Delta x_0} = 2\sqrt{\frac{h}{2M}}\sqrt[4]{\frac{2H}{g}} \quad \propto \quad \left(\frac{h}{M}\right)^{1/2}\left(\frac{H}{g}\right)^{1/4}.$$

For the given mass and height we get $\Delta x > 4\times10^{-16}$ m, smaller than size of nucleus.

QM – C1 SOLUTION

(a) We have

$$Y(\theta,\phi) = A\sin\theta\cos\phi = \frac{A}{2}\sin\theta\left(e^{i\phi}+e^{-i\phi}\right) = \frac{A}{2}\left(-\left(\frac{8\pi}{3}\right)^{\frac{1}{2}}Y_1^1 + \left(\frac{8\pi}{3}\right)^{\frac{1}{2}}Y_1^{-1}\right)$$

Simplify by using Dirac notation: $Y(\theta,\phi) \to |Y\rangle$, $Y_1^{\pm1} \to |Y_1^{\pm1}\rangle$

Then we have: $|Y\rangle = -A\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}(|Y_1^1\rangle - |Y_1^{-1}\rangle)$

Now, since: $\langle Y|Y\rangle = 1$, $\left\langle Y_1^{\pm1}\middle|Y_1^{\pm1}\right\rangle = 1$, and $\left\langle Y_1^{\pm1}\middle|Y_1^{\mp1}\right\rangle = 0$

We have: $1 = \langle Y|Y\rangle = A^2\frac{2\pi}{3}\left((\langle Y_1^1| - \langle Y_1^{-1}|)\,(|Y_1^1\rangle - |Y_1^{-1}\rangle)\right)$

$$= A^2\frac{2\pi}{3}(\langle Y_1^1|Y_1^1\rangle - \langle Y_1^{-1}|Y_1^{-1}\rangle)$$

$$= A^2\frac{4\pi}{3} \implies A = \sqrt{\frac{3}{4\pi}}$$

So: $|Y\rangle = -\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}(|Y_1^1\rangle - |Y_1^{-1}\rangle) = -\frac{1}{\sqrt{2}}(|Y_1^1\rangle - |Y_1^{-1}\rangle)$

(b) Since: $L_z|Y_l^m\rangle = \hbar m|Y_l^m\rangle$ and $L^2|Y_l^m\rangle = \hbar l(l+1)|Y_l^m\rangle$

We have: $L_z|Y\rangle = -\frac{1}{\sqrt{2}}(\hbar|Y_1^1\rangle + \hbar|Y_1^{-1}\rangle) = -\frac{\hbar}{\sqrt{2}}(|Y_1^1\rangle + |Y_1^{-1}\rangle)$

So : $\langle L_z\rangle = \langle Y|L_z|Y\rangle = \frac{\hbar}{2}\left((\langle Y_1^1| - \langle Y_1^{-1}|)\,(|Y_1^1\rangle + |Y_1^{-1}\rangle)\right) = \frac{\hbar}{2}(\langle Y_1^1|Y_1^1\rangle - \langle Y_1^{-1}|Y_1^{-1}\rangle) = \frac{\hbar}{2}(1-1) = 0$

Similarly: $L^2|Y\rangle = -\frac{1}{\sqrt{2}}(L^2|Y_1^1\rangle + L^2|Y_1^{-1}\rangle) = -\frac{\hbar^2}{\sqrt{2}}(2|Y_1^1\rangle + 2|Y_1^{-1}\rangle) = 2\hbar^2|Y\rangle$

So: $\langle L^2\rangle = \langle Y|L^2|Y\rangle = 2\hbar^2\langle Y|Y\rangle = 2\hbar^2$

QM – C2 SOLUTIONS

(a) To get the eigenvalues, one needs to solve H|ψ> = E|ψ>:

$$E_0\begin{pmatrix}1 & \lambda \\ \lambda & -1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = E\begin{pmatrix}x \\ y\end{pmatrix}$$

$$\begin{pmatrix}E_0 - E & E_0\lambda \\ E_0\lambda & -E_0 - E\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = 0$$

$$E = \pm E_0\sqrt{1+\lambda^2}$$

(b) In the unperturbed case, the eigenvectors are |U> and |D>, corresponding to eigenvalues of $+E_0$ and $-E_0$ respectively. To get to the second order, we write

$$E_U = E_0 + \langle U|H_V|U\rangle + \frac{\langle D|H_V|U\rangle^2}{E_0-(-E_0)} = E_0 + 0 + \frac{E_0{}^2\lambda^2}{2E_0} \quad = E_0\left(1+\frac{\lambda^2}{2}\right)$$

$$E_D = -E_0 + \langle D|H_V|D\rangle + \frac{\langle U|H_V|D\rangle^2}{-E_0-E_0} = -E_0 + 0 - \frac{E_0{}^2\lambda^2}{2E_0} \quad = -E_0\left(1+\frac{\lambda^2}{2}\right)$$

Expanding the exact answer from 1) in powers of λ we get $E = \pm E_0\left(1+\frac{\lambda^2}{2}-\frac{\lambda^4}{8}+...\right)$

which coincides with the perturbation theory answer up to λ^2.

(c) First, let us determine the eigenvectors of the system. They are given by equation

$$(E_0 - E)x + E_0\lambda y = 0.$$

Substituting results for E from (b) we get two vectors:

$$|\psi_1\rangle = \left(1-\frac{1}{8}\lambda^2\right)\begin{pmatrix}1 \\ \frac{1}{2}\lambda\end{pmatrix} = \left(1-\frac{1}{8}\lambda^2\right)|U\rangle + \frac{1}{2}\lambda|D\rangle$$

$$|\psi_2\rangle = \left(1-\frac{3}{8}\lambda^2\right)\begin{pmatrix}-\frac{\lambda}{2} \\ 1+\frac{\lambda^2}{4}\end{pmatrix} = -\frac{1}{2}\lambda|U\rangle + \left(1-\frac{1}{8}\lambda^2\right)|D\rangle$$

It is easy to show that |U> and |D> can be represented as

$$|U\rangle = \left(1-\frac{1}{8}\lambda^2\right)|\psi_1\rangle - \frac{1}{2}\lambda|\psi_2\rangle$$
$$|D\rangle = \frac{1}{2}\lambda|\psi_1\rangle + \left(1-\frac{1}{8}\lambda^2\right)|\psi_2\rangle$$

If a system was initially in a state $|\psi_k\rangle$, after a time t its wavefunction will be

$$|\psi_k(t)\rangle = |\psi_k(0)\rangle \times e^{-i\frac{E_k}{h}t}.$$

Therefore if the system was initially in state $|U\rangle$,

$$|\psi(t)\rangle = \left(1-\frac{1}{8}\lambda^2\right)|\psi_1\rangle e^{-i\frac{E_U}{h}t} - \frac{1}{2}\lambda|\psi_2\rangle e^{-i\frac{E_D}{h}t} =$$
$$= \left(\left(1-\frac{1}{8}\lambda^2\right)|U\rangle + \frac{1}{2}\lambda|D\rangle\right)e^{-i\frac{E_U}{h}t} + \left(-\frac{1}{8}\lambda^2|U\rangle - \frac{1}{2}\lambda|D\rangle\right)e^{-i\frac{E_D}{h}t}$$

The probability to find the system in the state $|D\rangle$ is $\langle D|\psi(t)\rangle^2$:

$$P(t) = \left|\frac{1}{2}\lambda\cdot e^{-i\frac{E_U}{h}t}\cdot\left(1-e^{-i\frac{E_D-E_U}{h}t}\right)\right|^2 = \frac{\lambda^2}{2}\left(1-\cos\frac{2E_0\left(1+\frac{\lambda^2}{2}\right)t}{h}\right)$$

QM - D1 SOLUTION

a) We need to add three angular momenta with $s_B = 1/2$, $s_C = 1$ and l to get a total angular momentum $s_A = 1/2$. Use the rule when adding angular momenta, j_1 and j_2 that $|j1 - j_2| \leq j \leq j_1 + j_2$. First add s_B and s_C.

$$|s_1 - s_2| \leq s \leq s_1 + s_2 \;\rightarrow\; 1/2 \leq s \leq 3/2$$

Possibilities are: $s = 1/2,\, 3/2$

Now add s and l to get s_A

$$|1/2 - l| \leq 1/2 \leq 1/2 + l \;\rightarrow\; l = 0,\, 1$$
$$|3/2 - l| \leq 1/2 \leq 3/2 + l \;\rightarrow\; l = 1,\, 2$$

Answer is $l = 0,\, 1,\, 2$

b) Since $l = 0$, we only need to consider the addition of s_B and s_C. From the given Clebsch-Gordon coefficients, we see that

$$|\,3/2, 1/2\,\rangle \;=\; \langle\,1, 1; 1/2, -1/2\,|\,3/2, 1/2\,\rangle\,|\,1, 1; 1/2, -1/2\rangle$$

$$+\,\langle\,1, 0; 1/2, 1/2\,|\,3/2, 1/2\,\rangle\,|\,1, 0; 1/2, 1/2\rangle$$

$$=\; \sqrt{1/3}\,|\,1, 1; 1/2, -1/2\rangle \;+\; \sqrt{2/3}\,|\,1, 0; 1/2, 1/2\rangle$$

Now the state that we're interested in is $|\,1/2, 1/2\,\rangle$. This state is orthogonal to $|\,3/2, 1/2\,\rangle$ so

$$|\,1/2, 1/2\,\rangle \;=\; \sqrt{2/3}\,|\,1, 1; 1/2, -1/2\rangle \;-\; \sqrt{1/3}\,|\,1, 0; 1/2, 1/2\rangle$$

So if the initial state is $|\,1/2, 1/2\,\rangle$, the probability for the final state to be in the state $|\,1, 0; 1/2, 1/2\rangle$ with the spin of particle B in the $+z$-direction is $1/3$.

Note that we have used:

$$\langle\,1, 1; 1/2, -1/2\,|\,3/2, 1/2\,\rangle \;=\; \sqrt{1/3} \qquad \langle\,1, 0; 1/2, 1/2\,|\,3/2, 1/2\,\rangle \;=\; \sqrt{2/3}$$

QM – D2   SOLUTION

To first order, the amplitude for transition to state |n’> is given by

$$a_{n'}(t) = \frac{-i}{\hbar}\int_{t_o}^{t} \langle n'|H_I|n\rangle e^{-i\left(E_{n'}-E_n\right)t/\hbar}\, dt$$

Where t_o is the onset of the perturbation H_I. The perturbation is the electric field: $-e\mathcal{E}_o\ e^{-t^2/\tau^2}$

with corresponding potential energy: $H_I = -ex\mathcal{E}_o\ e^{-t^2/\tau^2}$

Now, for the harmonic oscillator, the *x*-operator may be written as: $x_{op} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^{+})$

In terms of the raising and lowering operators:

$$a|n\rangle = n^{1/2}|n-1\rangle\ ;\ a^{+}|n\rangle = (n+1/2)^{1/2}|n+1\rangle$$

For the matrix element, with the initial state $|n\rangle = |2\rangle$ we have:

$$\langle n'|H_I|2\rangle = -e\left(\frac{\hbar}{2m\omega}\right)^{1/2}\mathcal{E}_o\ e^{-t^2/\tau^2}\langle n'|(a+a^{+})|2\rangle$$

The only non-zero terms will be $\langle 1|a|2\rangle = \sqrt{2}\langle 1|1\rangle = \sqrt{2}$ and $\langle 3|a^{+}|2\rangle = \sqrt{3}\langle 3|3\rangle = \sqrt{3}$.

So, there are transitions to two states:

$$a_1(t) = \frac{-i}{\hbar}\int_{-\infty}^{t} -e\left(\frac{\hbar}{2m\omega}\right)^{1/2}\sqrt{2}\mathcal{E}_o\ e^{-t^2/\tau^2}e^{-i\omega_o t}\, dt$$

$$a_3(t) = \frac{-i}{\hbar}\int_{-\infty}^{t} -e\left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\sqrt{3}\mathcal{E}_o\ e^{-t^2/\tau^2}e^{i\omega_o t}\, dt$$

From the given integral, we have

$$\int_{-\infty}^{\infty} e^{-t^2/\tau^2}e^{\pm i\omega_o t}\, dt = \tau\pi e^{-\tau^2\omega_o{}^2/4}$$

So: $P_{21} = |a_1(\infty)|^2 = \frac{2e^2\mathcal{E}^2\pi\tau^2}{2m\hbar\omega}e^{-\tau^2\omega_o{}^2/2}$ and $P_{23} = |a_3(\infty)|^2 = \frac{3e^2\mathcal{E}^2\pi\tau^2}{2m\hbar\omega}e^{-\tau^2\omega_o{}^2/2}$

No other transitions occur to first order.

(b) The value of *t* that maximizes this probability is given by $\frac{\partial P(\tau)}{\partial\tau} = 0.$
In both cases, $P(\tau)\ \sim\ \tau^2 e^{-\tau^2\omega_o{}^2/2}$

So: $\frac{\partial P(\tau)}{\partial\tau}\ \sim\ \left[2\tau e^{-\tau^2\omega_o{}^2/2} - \tau^3\omega_o{}^2 e^{-\tau^2\omega_o{}^2/2}\right] = 0$ which occurs when $2 - \tau^2\omega_o{}^2 = 0\ \Rightarrow$
$\tau = \frac{\sqrt{2}}{\omega}$