CM A Solution:

a) The muons are traveling close to the speed of light, so the time taken to reach the ground is

$$t = h/c = \frac{60 \times 10^3 \text{ m}}{3.0 \times 10^8 \text{ m s}^{-1}} = 2 \times 10^{-4} \text{ s}. \tag{1}$$

This is $(2 \times 10^{-4}\text{ s})/(1.5 \times 10^{-6}\text{ s}) = 133$ times larger than the half-life.

b) Evidently, the time to taken reach the ground in the rest frame of the muon is $3t_{1/2}$. One way to find $1-\beta$ is to calculate the relativistic interval, $\Delta\tau = ((c\Delta t)^2 - (\Delta x)^2)^{1/2}$, for the path of the muon between creation and reaching the ground and use the independence of the interval from the reference frame. In the rest frame of the muon, $\Delta\tau = 3ct_{1/2}$. In the laboratory frame, $\Delta\tau = ((h/\beta)^2 - h^2)^{1/2} = h(1/\beta^2 - 1)^{1/2}$. Equating the two expressions yields

$$h(1/\beta^2 - 1)^{1/2} = 3ct_{1/2} \tag{2}$$

$$\Rightarrow \frac{1-\beta^2}{\beta^2} = \left(\frac{3ct_{1/2}}{h}\right)^2 \tag{3}$$

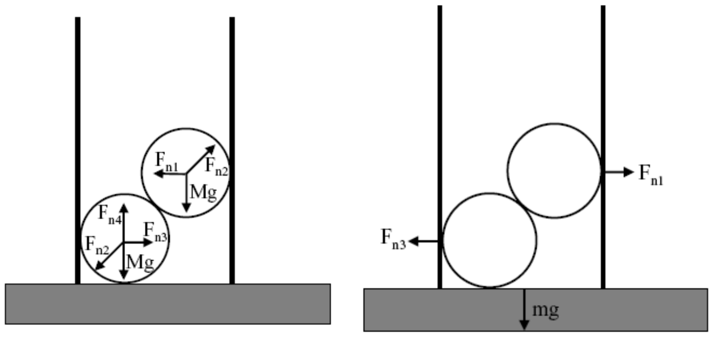
$$\Rightarrow \frac{(1-\beta)(1+\beta)}{\beta^2} = \left(\frac{3ct_{1/2}}{h}\right)^2. \tag{4}$$

Since $\beta \approx 1$, $1+\beta \approx 2$ and $\beta^2 \approx 1$. Then,

$$2(1-\beta) \approx \left(\frac{3ct_{1/2}}{h}\right)^2 \Rightarrow (1-\beta) = \frac{1}{2}\left(\frac{3ct_{1/2}}{h}\right)^2. \tag{5}$$

Plugging in numbers (not required), yields $1-\beta \approx 2.5 \times 10^{-4}$.

CM - B: Solution:



For the top ball, $Mg = \frac{F_{n_2}}{\sqrt{2}}$ and $F_{n_1} = \frac{F_{n_2}}{\sqrt{2}}$. Thus, $F_{n_1} = Mg$ and $F_{n_2} = \frac{Mg}{\sqrt{2}}$.

For the bottom ball, $F_{n_3} = \frac{F_{n_2}}{\sqrt{2}}$ and $Mg + \frac{F_{n_2}}{\sqrt{2}} = F_{n_4}$. Thus $F_{n_3} = Mg$ and $F_{n_4} = 2Mg$.

For equilibrium, the total torque has to be zero, therefor:

$$0 = F_{n_1}\left(R + \sqrt{2}R\right) - F_{n_3}R - mg\left(2R + \sqrt{2}R\right)/2$$

Substituting we get: $0 = M\left(1 + \sqrt{2}\right) - M - m\left(1 + 1/\sqrt{2}\right)$

Leading to: $m = 2M/\left(1 + \sqrt{2}\right)$.

CM - C1 Solution:

a) If r is the distance of a mass element dm from the center of mass, then the moment of inertia around the center of mass is

$$I \equiv \int r^2 dm. \tag{1}$$

Writing the integral as twice the integral from the center of the bar (its center of mass) to the end of the bar and expressing dm as $(M/L)dz$ yields

$$I = 2\int_0^{L/2} z^2 (M/L) dz = 2(M/L)\left[\frac{1}{3}z^3\right]_0^{L/2} = \frac{1}{12}ML^2. \tag{2}$$

b) The Lagrangian is the total kinetic energy minus the total potential energy, $L \equiv K - U$.

The total kinetic energy is the sum of the translational and rotational kinetic energy. Now

$$K_{tr} = \frac{1}{2}M(\dot{x}_{cm})^2 = \frac{1}{8}M(\dot{x}_1 + \dot{x}_2)^2, \tag{3}$$

which uses the expression $(x_1 + x_2)/2$ for the coordinate of the center of mass of the bar, x_{cm}. Let θ be the angle that the bar makes with the z-axis. Then $\tan(\theta) = (x_2 - x_1)/L$ and, for small deviations from equilibrium ($x_1 \ll L$ and $x_2 \ll L$), $\theta = (x_2 - x_1)/L$. Then

$$K_{rot} = \frac{1}{2}I(\dot{\theta})^2 = \frac{1}{2}\left(\frac{1}{12}ML^2\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2 = \frac{1}{24}M(\dot{x}_2 - \dot{x}_1)^2. \tag{4}$$

The total potential energy is the sum of the gravitational potential energy and the potential energy stored in the springs. The first is simply

$$U_g = Mgx_{cm} = \frac{1}{2}Mg(x_1 + x_2). \tag{5}$$

The energy stored in spring 1 is

$$U_{sp} = \frac{1}{2}k(x_0 - x_1)^2 - \frac{1}{2}k(x_0)^2, \tag{6}$$

where x_0 is the position where the spring exerts no force and the (arbitrary) zero-point for the potential energy has been defined to be that at the equilibrium position of the bar. This definition slightly simplifies the final expression for the total potential energy. The expression for U_{sp} can be simplified to

$$U_{sp} = -kx_0x_1 + \frac{1}{2}k(x_1)^2 = -\frac{1}{2}Mgx_1 + \frac{1}{2}k(x_1)^2. \tag{7}$$

The last step uses the result that the equilibrium position of the bar is defined by $kx_0 = Mg/2$. Thus, the total potential energy is

$$U = \frac{1}{2}Mg(x_1 + x_2) - \frac{1}{2}Mgx_1 + \frac{1}{2}k(x_1)^2 - \frac{1}{2}Mgx_2 + \frac{1}{2}k(x_2)^2 = \frac{1}{2}k(x_1)^2 + \frac{1}{2}k(x_2)^2. \tag{8}$$

Combining equations 3, 4, and 8 yields the Lagrangian:

$$L = \frac{1}{8}M(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{24}M(\dot{x}_2 - \dot{x}_1)^2 - \frac{1}{2}k(x_1)^2 - \frac{1}{2}k(x_2)^2. \tag{9}$$

c) To find the frequencies of the normal modes, start by deriving the equations of motion.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} &= 0 && (10)\\ \frac{d}{dt}\left(\frac{1}{4}M\left(\dot{x}_1 + \dot{x}_2\right) - \frac{1}{12}M\left(\dot{x}_2 - \dot{x}_1\right)\right) + kx_1 &= 0 && (11)\\ \frac{1}{3}M\ddot{x}_1 + \frac{1}{6}\ddot{x}_2 + kx_1 &= 0. && (12) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= 0 && (13)\\ \frac{1}{6}M\ddot{x}_1 + \frac{1}{3}\ddot{x}_2 + kx_2 &= 0. && (14) \end{aligned}$$

The normal modes are solutions of the form $x_1 = A\sin(\omega t)$ and $x_2 = B\sin(\omega t)$. Plugging these solutions into equations 12 and 14 yields

$$M\omega^2\left(\frac{1}{3}A + \frac{1}{6}B\right) = kA \quad \text{and} \quad M\omega^2\left(\frac{1}{6}A + \frac{1}{3}B\right) = kB. \tag{15}$$

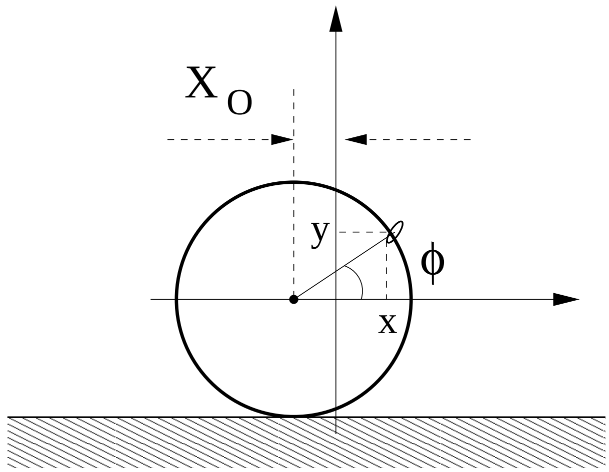
Using these two equations to eliminate $k/(M\omega^2)$ and then solving for B/A produces the equation $(B/A)^2 = 1$, which has the solutions $B = \pm A$. Plugging these solutions into either equation 12 or 14 then yields the frequencies

$$\omega_+ = \left(\frac{2k}{M}\right)^{1/2} \quad \text{and} \quad \omega_- = \left(\frac{6k}{M}\right)^{1/2}. \tag{16}$$

d) The normal mode corresponding to ω_+ has the bar moving up and down while remaining horizontal. That corresponding to ω_- has the center of mass of the bar remain stationary while the two ends oscillate up and down exactly out of phase.

CM - C2 Solution

We choose the origin of the coordinate system at the center of mass. Let (x, y) and X_O be the bead coordinates and the horizontal coordinate of the center of hoop, respectively.



$$x = X_O + R\ \sin\phi\ , \qquad y = R\ \cos\phi\ .$$

Since the center of mass of the system remains at rest,

$$m\,x + M\,X_O = 0\ ,$$

we have

$$x\left(1 + \frac{m}{M}\right) = R\ \sin\phi\ , \qquad y = R\ \cos\phi\ ,$$

and, hence,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{with} \qquad a = \frac{R}{1 + \frac{m}{M}}\ , \quad b = R\ .$$

This equation defines an ellipse with semi-axes a and b, and with its center at the center of mass. The bird trajectory is the lower arc of this ellipse stretched between the points (x_0, y_0) and $(-x_0, y_0)$, where (x_0, y_0) is an initial position of the bead.

CM - D1 Solution

Part I:

(a) The equation of motion

$$m\,\ddot{x} = -g\ \mathrm{sign}(x)$$

describes uniform acceleration $-g$ for positive x, and $+g$ for negative x. The trajectory thus is the juxtaposition of parabolas

$$x(t) - x_0 = -\frac{g}{2m}\,(t - t_0 + T/4)^2 \qquad \text{for} \quad -T/2 \le t - t_0 \le 0\,,$$

$$x(t) + x_0 = +\frac{g}{2m}\,(t - t_0 - T/4)^2 \qquad \text{for} \quad 0 \le t - t_0 \le T/2\,,$$

periodically repeated with the period

$$T = 4\sqrt{\frac{2mx_0}{g}}\,.$$

Here $x_0 \ge 0$ is the amplitude of the oscillations.

(b) The amplitude x_0 relates to the energy E as

$$E = g\,x_0\,,$$

hence

$$T(E) = \frac{4\sqrt{2mE}}{g}\,.$$

(c) Since

$$E = \frac{p^2}{2m} + g\,|x|$$

we have

$$p = \pm\sqrt{2m(E - g\,|x|)}$$

where the sign changes at the turning points $|x| = x_0$. Elementary calculation yields

$$S = \int_{t=t_0}^{t_0+T} p dx = 4\int_0^{x_0} \sqrt{2mg(x_0 - x)}\,dx = \frac{8\sqrt{2m}}{3g}\,E^{3/2} \qquad (P1.1)$$

Part II.

(a) Now the equation of motion is

$$\frac{d}{dt}\left(\frac{m\,\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + g\ \mathrm{sign}(x) = 0\,.$$

As conservative system, it has the first integral

$$E = \frac{m}{\sqrt{1-\dot{x}^2}} + g\,|x|\,,$$

which allows one to integrate the equation of motion:

$$(X-x)^2-(t-t_0)^2=R^2 \qquad \text{for} \quad x\geq 0\,,$$

$$(X+x)^2-(t-t_0)^2=R^2 \qquad \text{for} \quad x\leq 0\,.$$

Here

$$R=\frac{m}{g}\,, \qquad X=\frac{E}{g}\,,$$

and t_0 is an arbitrary time. For real motion $E\geq m$, hence

$$X\geq R\,.$$

Thus, the trajectory is the periodic (with the period T) extension of the piecewise hyperbolic cycle

$$x(t)=+X-\sqrt{R^2+(t-t_0+T/4)^2} \qquad \text{for} \quad -T/2\leq t-t_0\leq 0\,,$$

$$x(t)=-X+\sqrt{R^2+(t-t_0-T/4)^2} \qquad \text{for} \quad 0\leq t-t_0\leq T/2\,.$$

Here

$$T=4\sqrt{X^2-R^2}$$

so that $x(t_0)=0$.

(b) Since $X=E/g$ we have

$$T(E)=\frac{4}{g}\sqrt{E^2-m^2}\,.$$

When $E=m+\Delta E$ with $\Delta E<<m$, this reduces to the result in the Part I, with ΔE replacing E.

(c) In terms of the relativistic momentum

$$p=\frac{m\,\dot{x}}{\sqrt{1-\dot{x}^2}}$$

we have

$$E=\sqrt{m^2+p^2}+g\,|x|\,.$$

The integral

$$\int_{t=-T/2}^{T/2} p(t)\,dx(t)=2\int_0^{T/2} p(t)\,dx(t)$$

is easy to evaluate with the change of variables

$$t-t_0-T/4=R\,\sinh\tau \qquad \text{for} \quad 0\leq t-t_0\leq T/2\,,$$

so that

$$X-x=R\,\cosh\tau\,, \qquad p=-m\,\sinh\tau\,.$$

When $t - t_0$ changes from 0 to $T/2$, τ changes from $-\theta$ to θ, where θ is determined from

$$R\,\sinh\theta = T/4 = \frac{1}{g}\,\sqrt{E^2 - m^2}\,,$$

or

$$E = m\,\cosh\theta\,. \tag{P1.2}$$

The integration yields

$$S = \frac{2m^2}{g}\int_{-\theta}^{\theta} d\tau\,\sinh^2\tau = \frac{m^2}{g}\,[\sinh 2\theta - 2\theta]\,. \tag{P1.3}$$

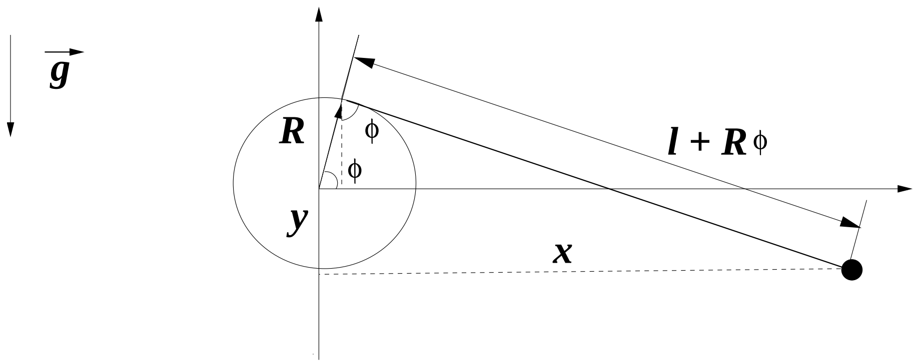
Together with (P1.2) this yields parametric representation of $S(E)$.

In the non-relativistic limit $\Delta E = E - m << m$ we have $\theta << 1$, whence

$$\Delta E \approx \frac{m}{2}\,\theta^2\,, \qquad S \approx \frac{4m^2}{3g}\,\theta^3\,,$$

and (P1.2),(P1.3) reduce to (P1.1).

CM - D2 Solution



Let ϕ be an angle between the string and the vertical axis,

$$x = (l + R\phi)\sin(\phi) + R\,\cos(\phi)\,, \qquad y = -(l + R\phi)\cos(\phi) + R\,\sin(\phi)$$

and

$$\dot{x} = \dot{\phi}\,(l + R\phi)\,\cos(\phi)\,, \quad \dot{y} = \dot{\phi}\,(l + R\phi)\,\sin(\phi)\ .$$

The kinetic, potential energies and the Lagrangian are given by,

$$\begin{aligned} T &= \frac{m}{2}\,(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\,\dot{\phi}^2(l + R\phi)^2 \\ U &= mg\,y = -mg\big(\,(l + R\phi)\cos(\phi) - R\,\sin(\phi)\,\big) \\ L &= \frac{m}{2}\,\dot{\phi}^2(l + R\phi)^2 + mg\big(\,(l + R\phi)\cos(\phi) - R\,\sin(\phi)\,\big)\ . \end{aligned}$$

The equation of motion reads explicitly,

$$\frac{d}{dt}\,\dot{\phi}(l + R\phi)^2 = \dot{\phi}^2 R(l + R\phi) - g(l + R\phi)\ \sin(\phi)\ ,$$

or

$$\ddot{\phi} = -\big(\dot{\phi}^2 R + g\,\sin(\phi)\,\big)/(l + R\phi)\ .$$

The energy of the pendulum is a conserved quantity,

$$E = \dot{\phi}\,\frac{\partial L}{\partial \dot{\phi}} - L = \frac{m}{2}\,\dot{\phi}^2(l + R\phi)^2 - mg\big(\,(l + R\phi)\cos(\phi) - R\,\sin(\phi)\,\big) = const\ ,$$

and, hence,

$$dt = \pm\frac{(l + R\phi)\,d\phi}{\sqrt{2E/m + 2g\big(\,(l + R\phi)\cos(\phi) - R\,\sin(\phi)\,\big)}}\ .$$

The period is given by the integral

$$T = \sqrt{\frac{2R}{g}} \int_{\phi_1}^{\phi_2} \frac{(\frac{l}{R} + \phi)\, d\phi}{\sqrt{\frac{E}{mgR} + (\frac{l}{R} + \phi)\cos(\phi) - \sin(\phi)}} \ ,$$

where $\phi_{1,2}$ are the turning points, i.e., two roots of the equation

$$\sin(\phi) - \left(\frac{l}{R} + \phi\right) \cos(\phi) = \frac{E}{mgR} \ .$$