Rutgers - Physics Graduate Qualifying Exam
Quantum Mechanics – January 18, 2008

QA

A free particle of mass m moving in one dimension is known to be in the initial state:

$$\Psi(x,t=0) = A\sin(k_o x)$$

(a) [3 pts.] What is $\Psi(x,t)$ at a later time t?

(b) [4 pts.] What is the probability distribution that a measurement of the momentum of this particle at this time t will find a value p?

(c) [3 pts.] If p is measured at $t = t_o$ seconds and the value $\hbar K_o$ is found, then what would be $\Psi(x,t)$ for times $t > t_o$?

QB

A quantum mechanical particle of mass m is moving in a cubical box of length L ($0 \le x, y, z \le L$).

(a) [2 pts.] What is the energy and normalized wave function of the ground state?

One wall of the box is suddenly moved from $x = L$ to $x = 4L$.

(b) [3 pts.] What is the energy and wave function in the ground and first excited states of a particle in the enlarged box?

(c) [5 pts.] What is the probability that the system which was in the ground state of the cube is still in the ground state of the enlarged box?

Useful integral: $\int(\sin mx \cdot \sin nx)dx = \dfrac{\sin(m-n)x}{2(m-n)} - \dfrac{\sin(m+n)x}{2(m+n)}$, where $(m^2 \neq n^2)$

QC1

Prove the virial theorem (average kinetic energy = average potential energy) for the n^{th} eigenstate of the simple harmonic oscillator.

QC2

Consider a particle of mass m in three dimensions in the following potential:

$$V(r) = \begin{cases} -V_o \\ 0 \end{cases} \text{ for } \begin{matrix} r < a \\ r > a \end{matrix}$$

where $V_o > 0$.

(a) [3 pts.] Recall that the energy levels in any central potential can be labeled by radial, n_r, and angular momentum, l, quantum numbers. Show that for fixed n_r the energy $E_{n_r l}$ of bound states (when they exist) increases as the angular momentum l is increased.

(b) [4 pts.] What is the condition on the parameters V_o, a and m such that there is at least one bound state?

(c) [3 pts.] Determine the number of s-states (*i.e.* bound states with $l = 0$).

QD1

Recall that the kinetic energy of a plane rotator moving in the xy-plane is:

$$\frac{\hbar^2 l_z^2}{2I}$$

where l_z is the z-component of the angular momentum operator. Suppose the rotator has an electric dipole moment $\boldsymbol{d}$ and a strong electric field $\boldsymbol{E}$ in the rotation plane ($dE >> \hbar^2 / I$) is applied. Determine approximately the low lying energy spectrum. What is the condition of applicability of your approximation?

Hint: Convince yourself that in a strong electric low energy states are localized in the region $|\phi| << 1$, where ϕ is the angle between $\boldsymbol{d}$ and $\boldsymbol{E}$. Then, you can expand the interaction energy of the rotator with the field in ϕ to the lowest nontrivial order to obtain a familiar Hamiltonian.

QD2

This problem explores the basic ideas behind the phenomenon of magnetic circular dichroism.

The eigenfunctions for non-interacting electrons in a hydrogenic potential can be written as a product wave function of the spatial and spin degrees of freedom as follows:

$$\left|n,l,m,\uparrow\right\rangle = \psi_{nlm}(r,\theta,\phi)\left|\uparrow\right\rangle = AR_{n,l}(r)Y_l^m(\theta,\phi)\left|\uparrow\right\rangle$$

where $R_{n,l}(r)$ is a Leguerre polynomial and $Y_l^m(\theta,\phi)$ a spherical harmonic, $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ indicate the component of the electron spin parallel or antiparallel to the quantization axis, and A is a normalization constant. The quantum numbers are: n, the principal quantum number, l the quantum number of the orbital angular momentum operator L, and m the quantum number of the operator for the z-component of the orbital angular momentum L_z. In this problem, consider the 2*p* level, *i.e.*, $n = 2$, $l = 1$.

(a) [1 pts.] What are the possible values of m?

(b) [1 pts.] The total angular momentum operator is $J = L + S$ with quantum numbers j, and the operator for the z-component of total angular momentum is $J_z = L_z + S_z$ with quantum numbers m_j. What are the possible values for j and their associated m_j's?

(c) [3 pts.] Find the simultaneous eigenfunctions of the operators J and J_z as linear combinations of the set $\left|2,1,m,\uparrow\right\rangle$ and $\left|2,1,m,\downarrow\right\rangle$ with appropriate values of m.

The probability for optical transitions are governed by the dipole matrix element

$$\left|M_{fi}\right|^2 = \left|\left\langle\Psi_f\right|\hat{\varepsilon}\bullet\vec{r}\left|\Psi_i\right\rangle\right|^2$$

Suppose the atom under consideration is spin polarized, so that optical transitions can only occur between initial states that are the eigenstates from part (c) to final states of the (degenerate) 3*d* levels of the form: $\left|3,2,m,\downarrow\right\rangle$.

(d) [2 pts.] Consider dipole transitions excited by linearly polarized light where $\hat{\varepsilon} = \hat{z}$. Show that such excitations obey the selection rules: $l_f = l_i \pm 1$ and $m_f = m_i$.

(e) [3 pts.] Now, consider dipole transitions excited by circularly polarized light, where $\hat{\varepsilon} = (\hat{x} \pm i\hat{y})$ for left and right circular polarization, respectively. Show that the selection rules now become $l_f = l_i \pm 1$ and $m_f = m_i \pm 1$. Show that the asymmetry $[(I_{\text{left}} - I_{\text{right}})_{j'} / (I_{\text{left}} - I_{\text{right}})_j]$ is -1, where I is the intensity of the transition.

The asymmetry in part (e) is called magnetic circular dichroism and, since optical transitions energies are different for different elements, is used to identify the atomic source of ferromagnetism in many complex materials.

Possibly useful: $J_\pm\left|j,m_j\right\rangle = \hbar\sqrt{j(j+1)-m_j(m_j\pm 1)}\left|j,m_j\pm 1\right\rangle$; similar for L and S.