Solution 1:

$mg=GMm/r^2$, so $GM=gR^2$.

At the equator, $mV^2/R=GMm/R^2 - mg' = mg - 2mg/3 = mg/3$.

Hence, $g = 3V^2/R$.

Potential energy of a particle at the surface is given by: $-GMm/R$, and the total energy of a particle at the pole, traveling with speed u is:

$E = mu^2/2 - GMm/R = mv^2/2 - GMm/r$ for any r and v. This is equal to 0 for escape, i.e. $r \to \infty$ as $v \to 0$.

Thus, $u=v_{esc}$ when $mu^2/2 = GMm/R$.

So, $v_{esc}^2 = 2GM/R = 2gR = 6V^2$, and

$v_{esc} = \sqrt{6}\, V$.

Solution Due to the rotation of the planet, the angular momentum of the stone at the moment of release is $(R+h)^2\omega_e$. This is a conserved quantity, so that if we measure the azimuthal angle ϕ of the stone from its position at the moment or release, one has

$$\dot{\phi} = \frac{(R+h)^2}{(R+y)^2}\omega_e \approx \omega_e + 2\omega_e \frac{h-y}{R}, \tag{1}$$

where $R+y$ is the distance to the stone from the center of the planet. We have used the fact that $h \ll R$ in obtaining the second (approximate) equality. Since the last term is already of order ω_e, we may simply use $y = h - \frac{1}{2}gt^2$ to evalutate it, where t is the time since the stone was dropped. Therefore

$$R(\dot{\phi} - \omega_e) = \omega_e g t^2. \tag{2}$$

Since $\dot{\phi} > \omega_e$ for $t > 0$, the stone will land due east of the plumb line. The distance d away from the plumb line upon landing will be

$$d \equiv \int_0^{t_0} dt\, R(\dot{\phi} - \omega_e) = \tfrac{1}{3}\omega_e g t_0^3 = \frac{\omega_e g}{3}\left(\frac{2h}{g}\right)^{\frac{3}{2}} = \frac{\omega_e}{3}\sqrt{\frac{8h^3}{g}}, \tag{3}$$

where t_0 is the value of t when the stone hits the ground. Putting in values gives an order of magnitude of $d \sim 10$ cm.

Solution:

a) At equilibrium, $-dV/dx = 0$, so $[c(a^2-x^2)]/(x^2+a^2)^2 = 0$. Thus, $x_1 = +a$, $x_2 = -a$.

Now, $d^2V/dx^2 = 2cx(x^2-3a^2)/(x^2+a^2)^3$, so x_1 is unstable ($d^2V/dx^2 <0$), x_2 is stable.

For small oscillations, let $x = -a + x'$, where $x' << a$. Then the equation of motion is:

$m\, d^2x/dt^2 = - cx'(2a-x')/[(x'-a)^2 + a^2]^2 \approx - cx'/2a^3$.

So, $T = 2\pi/\omega = 2\pi \sqrt{(2ma^3/c)} = 2\pi a \sqrt{(2ma/c)}$.

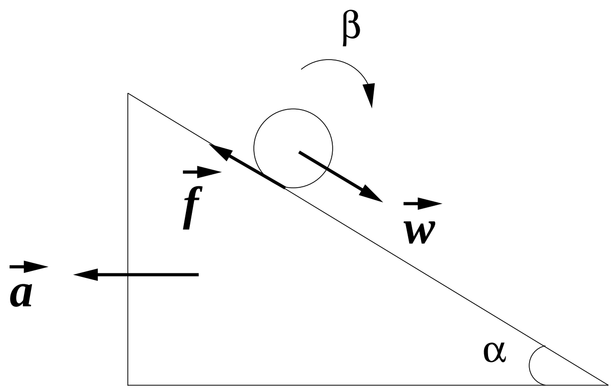
b) Starting at $x = -a$, the total energy of the particle is:

$E=mv^2/2 + V(-a) = mv^2/2 – c/2a$.

1) For confinement, $E<0$, so $v<\sqrt{(c/ma)}$.
2) At $-\infty$, $V=0$, so $E>0$ is the requirement for escape to $-\infty$. So, $v>\sqrt{(c/ma)}$.
3) To escape to $+\infty$, the particle must pass through $x_1 = +a$, where the potential energy is a maximum. Thus, $E > V(a) = c/2a$, so $v>\sqrt{(2c/ma)}$.

Solution

Elementary solution



Let w be the **magnitude** of an acceleration of the ball with respect to the wedge and a be the **magnitude** of an acceleration of the wedge in the laboratory system. Also let f be the **magnitude** of a force of friction. Then the projection of an acceleration of the ball on the direction parallel to the incline plane in the laboratory system is $w - a\cos\alpha$ and Newton's second law for this component gives

$$m(w - a\cos\alpha) = -f + mg\sin\alpha \ .$$

The projections of an acceleration of the ball on the horizontal plane in the laboratory system is $w\cos\alpha - a$, so the condition that the center of mass of the whole system does not move in the horizontal direction reads as follows:

$$m(w\cos\alpha - a) = Ma \ .$$

Since the ball is rolling down without slipping, an angular acceleration of the ball is given by

$$\beta = \frac{w}{R} \ ,$$

and satisfies the equation of motion:

$$I\beta = fR \ .$$

Hence

$$w - a\cos\alpha = -\frac{I}{mR^2}\, w + g\ \sin\alpha \ , \qquad w\cos\alpha = \left(1 + \frac{M}{m}\right) a \ , \tag{1}$$

and

$$a = \frac{g\ \sin\alpha\cos\alpha}{(1+\frac{M}{m})(\frac{I}{mR^2}+1) - \cos^2\alpha} \ .$$

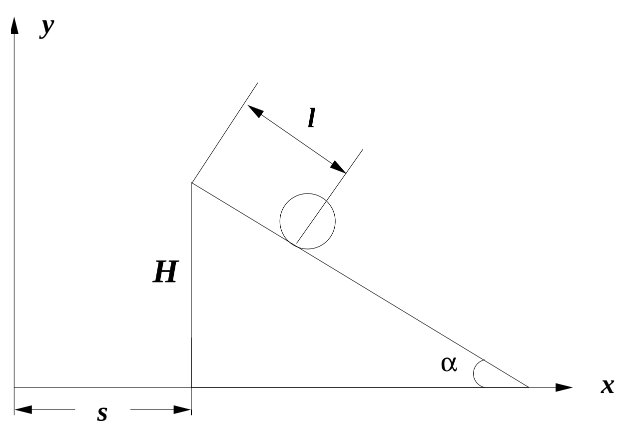
The moment of inertia is given by

$$I = \frac{m}{\frac{4}{3}\pi R^3} \int_0^R \mathrm{d}r r^2 \int_0^{2\pi} \mathrm{d}\theta \sin\theta \int_0^{\pi} \mathrm{d}\phi \; r^2 \sin^2\theta = \frac{2}{5} \; mR^2 \; .$$

Finally,

$$a = \frac{g \, \sin\alpha \cos\alpha}{\frac{7}{5} \, (1 + \frac{M}{m}) - \cos^2\alpha} \; .$$

Lagrange's approach



Introduce the generalized coordinates s and l as shown in Fig. The coordinates (x_0, y_0) of a center the ball are given by

$$x_0 = s + l\cos\alpha + R\sin\alpha \, , \quad y_0 = H - l\sin\alpha \; . \tag{2}$$

Since there is no slipping, the angular velocity of the ball is given by

$$\omega = \frac{\dot{l}}{R} \; . \tag{3}$$

The Lagrangian reads as follows

$$\begin{aligned} L &= \frac{M\dot{s}^2}{2} + \frac{m}{2} \left[(\dot{s} + \dot{l}\cos\alpha)^2 + \dot{l}^2 \sin^2\alpha \right] + \frac{I\dot{l}^2}{2R^2} + mgl\sin\alpha = \\ &= \frac{M\dot{s}^2}{2} + \frac{m}{2} \left(\dot{s}^2 + \dot{l}^2 + 2\dot{s}\dot{l}\cos\alpha \right) + \frac{I\dot{l}^2}{2R^2} + mgl\sin\alpha \; . \end{aligned}$$

Lagrange's equations coincide with Eqs.(1) provided $\ddot{s} = -a$, $\ddot{l} = w$.

Solution We have

$$T = \tfrac{1}{2}\left[M\dot{x}^2 + (m\dot{x}_1^2 + m\dot{y}_1^2)\right], \tag{1}$$

and

$$V = mgy_1, \tag{2}$$

where

$$x_1 = x + b\sin\theta \approx x + b\theta, \tag{3}$$

and

$$y_1 = b(1-\cos\theta) \approx \tfrac{1}{2}b\theta^2. \tag{4}$$

It is clear by inspection that there is no net horizontal force on the composite body consisting of both masses and the connecting string, so that the x component of the center of mass

$$x_c = \frac{Mx + mx_1}{M+m} \tag{5}$$

is not accelerated. In other words, there is no restoring force to the x component of center of mass motion, so that one normal mode has zero frequency, and simply consists of the motion $x = x_1 = x_0 + v_0 t$, where x_0 and v_0 are constants. To find the other mode, we can without loss in generality set $x_c = 0$, which implies that

$$x = -\frac{m}{M}x_1. \tag{6}$$

Use of (6) in (1) and (3), and then expressing T in terms of θ instead of the x's gives

$$T \approx \tfrac{1}{2}\mu b^2\dot{\theta}^2 \tag{7}$$

where μ is the reduced mass $Mm/(M+m)$. In writing (7), the term involving $\dot{y}$ is neglected, because it is higher order than quadratic in small excursions (using Eq. 4). The potential energy is similarly expanded as

$$V \approx \tfrac{1}{2}mgb\dot{\theta}^2. \tag{8}$$

The equation of motion is now easily found by either using $\dot{T} + \dot{V} = 0$ or by using Lagrangian techiques:

$$\ddot{\theta} + \frac{mg}{\mu b}\theta = 0, \tag{9}$$

so that the angular vibrational frequency of this mode is given by

$$\omega^2 = \frac{mg}{\mu b}. \tag{10}$$

The motion of the two masses in this mode is given by Eq. (6).

If one did not recognize the constant of motion that the above quick solution, the problem is soluble by standard methods. In that case, the kinetic energy for small displacements is

$$T = \tfrac{1}{2} M\dot{x}^2 + \tfrac{1}{2} m(x + b\dot{\theta})^2 \tag{11}$$

from which it is simple to use Lagrangian techniques to obtain

$$\ddot{x} + b\ddot{\theta} + g\theta = 0, \tag{12}$$

$$(M + m)\ddot{x} + mb\ddot{\theta} = 0, \tag{13}$$

which are straightforwardly solved to yield results results identical to those obtained above. Overall this method is more laborious, however.

Solution

Let $X_1,\ X_2$ and x be coordinates of the heavy particles and the light one respectively. For $X_1 < x < X_2$ the potential energy of the system reads,

$$U = (x - X_1)\, f + (X_2 - x)\, f = (X_2 - X_1)\, f \ .$$

Hence the "electron" moves freely between the "atoms". Under the condition

$$|\dot{X}_{1,2}| \ll |\dot{x}|$$

the quantity

$$\oint \frac{dx}{2\pi}\, p(x) \propto |\dot{x}|\, (X_2 - X_1) = C = const$$

is an adiabatic invariant. Combining this equation with the energy conservation law,

$$\frac{M}{2}\, (\, \dot{X}_1^2 + \dot{X}_2^2\,) + \frac{m\dot{x}^2}{2} + (X_2 - X_1)\, f = E = const \ ,$$

and assuming that the center mass of the system is at rest

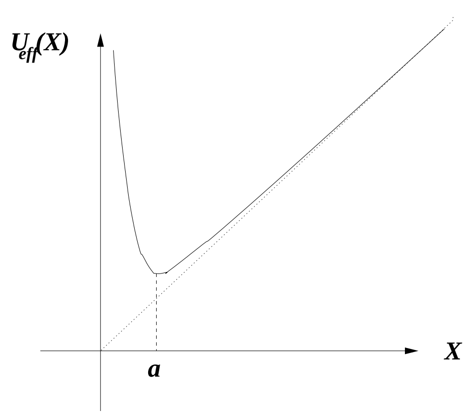
$$M\, (\dot{X}_2 + \dot{X}_1) + m\ \dot{x} = 0 \ ,$$

one obtains,

$$\frac{M\dot{X}^2}{4} + \left(1 + \frac{m}{2M}\right) \frac{mC^2}{2\, X^2} + X\, f = E \ , \qquad \text{where} \qquad X = X_2 - X_1 \ .$$

Since $m \ll M$ we can neglect the term $\propto m/M$. We have derived the equation which coincides with the energy conservation law for the $1D$ particle of mass $M/2$ moving in the effective potential

$$U_{eff}(X) = \frac{mC^2}{2\, X^2} + X\, f \ .$$



The stable equilibrium condition

$$\frac{dU_{eff}}{dX}\Big|_{X=a} = -\frac{mC^2}{a^3} + f = 0 \ ,$$

relates the constant C with the size of the "ion". Then

$$\frac{d^2U_{eff}}{dX^2}\Big|_{X=a} = \frac{3mC^2}{a^4} = \frac{3f}{a} \ ,$$

and the frequency of small oscillations is given by

$$\omega^2 = \frac{2}{M}\ \frac{d^2U_{eff}}{dX^2}\Big|_{X=a} = \frac{6f}{Ma} \ ,$$

Alternative Solution

Since the heavy particles move much more slowly than the light particles, we may approximately solve the problem by calculating the average force between the heavy particles at a fixed position, and use that to determine the motion of the heavy particles. This average force has a repulsive component F_R produced by the collisions of the light particle with each heavy particle. We note that the speed v of the light particle for a particular spacing X between the heavy particles is independent of time, because the collisions are elastic, and the contributions to the long range force from each heavy particle cancels. The outward momentum transfer per collision is $2mv$, with a time of $2X/v$ between collisions with the same heavy particle. Thus F_R is the quotient of these two quantities, $F_R = mv^2/X$. The dependence of v on X may be obtained by the work–kinetic energy theorem

$$F_R dX = -d\left(\frac{1}{2}mv^2\right) = -mvdv.$$

On substitution of the value of F_R above, this equation simplifies to $vdX = -Xdv$, that is $d(Xv) = 0$ and $v = C/X$, where C is a constant. Thus the repulsive force is given by

$$F_R = \frac{mC^2}{X^3}.$$

The net average force is then

$$F(X) = F_R - f = \frac{mC^2}{X^3} - f = f\left(\frac{a^3}{X^3} - 1\right) \approx -\frac{3f}{a}(X - a)$$

where the second equality uses the equilibrium condition $F(a) = 0$, and the leading term in $(X - a)$ determines the force constant $k = 3f/a$. The oscillation frequency ω is thus given by

$$\omega^2 = \frac{k}{\mu} = \frac{6f}{Ma},$$

where μ is the reduced mass, here equaling $M/2$.