Useful Formulae and Integrals

$$\int_0^\infty x^n e^{-x}\, dx = n! \qquad \text{(n = positive integer)}$$

$$\int_0^\infty e^{-\beta x^2}\, dx = \frac{1}{2}\, (\pi/\beta)^{1/2}$$

$$\int_0^\infty x\, e^{-\beta x^2}\, dx = \frac{1}{2\beta}$$

$$\int_0^\infty x^2\, e^{-\beta x^2}\, dx = \frac{1}{4}\, (\pi/\beta^3)^{1/2}$$

$$\int_0^\infty x^3\, e^{-\beta x^2}\, dx = \frac{1}{2\beta^2}$$

Stirling's approximation:

$$\log_e (n!) \simeq n \log_e n - n \qquad (n >> 1)$$

$$n! = \sqrt{2\pi\, n}\; n^n \exp\left(-n + \frac{1}{12n} + \circ\left(\frac{1}{n^2}\right)\right)$$

$$(a+b)^c = \sum_{r=0}^{c} a^r\, b^{(c-r)} \frac{c!}{r!(c-r)!}$$

Spherical Harmonics Y_ℓ^m

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,\cos\theta$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\phi}$$

$$Y_2^2(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{2i\phi}$$

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\, e^{i\phi}$$

$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\, e^{-i\phi}$$

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{-2i\phi}$$

Legendre Polynomials $P_\ell(z)$

$$P_o(z) = 1$$

$$P_1(z) = z$$

$$P_2(z) = \tfrac{1}{2}(3z^2 - 1)$$

$$P_3(z) = \tfrac{1}{2}(5z^3 - 3z)$$

$$Y_\ell^o(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell(\cos\theta)$$

The Bessel and Neumann functions $Z_n(x) = \begin{cases} J_n(x) \\ Y_n(x) \end{cases}$ are solutions of Bessel's equation

$$\frac{d^2 Z_n(x)}{dx^2} + \frac{1}{x}\frac{dZ_n(x)}{dx} + \left(1 - \frac{n^2}{x^2}\right) Z_n(x) = 0$$

$e = 4.80 \times 10^{-10}$ esu $= 1.60 \times 10^{-19}$ Coulomb

$c = 3.00 \times 10^{10}$ cm/sec $= 3.00 \times 10^{8}$ m/sec

$\hbar = 1.05 \times 10^{-27}$ erg sec $= 1.05 \times 10^{-34}$ J sec

$m_e = 9.11 \times 10^{-28}$ g $= 9.11 \times 10^{-31}$ kg

$m_p = 1.67 \times 10^{-24}$ g $= 1.67 \times 10^{-27}$ kg

$N_o = 6.02 \times 10^{23}$ particles/mole

$k_B = 1.38 \times 10^{-23}$ J K^{-1} $= 1.38 \times 10^{-16}$ ergK^{-1}

$a_o = 0.529 \times 10^{-8}$ cm

$\varepsilon_o \cong \dfrac{1}{4\pi\times 9\times 10^9}\,\dfrac{\text{Farad}}{\text{m}}$

$\mu_o = 4\pi \times 10^{-7}\,\dfrac{\text{Tesla}\cdot\text{m}}{\text{Ampere}}$

VECTOR OPERATIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

CYLINDRICAL COORDINATES

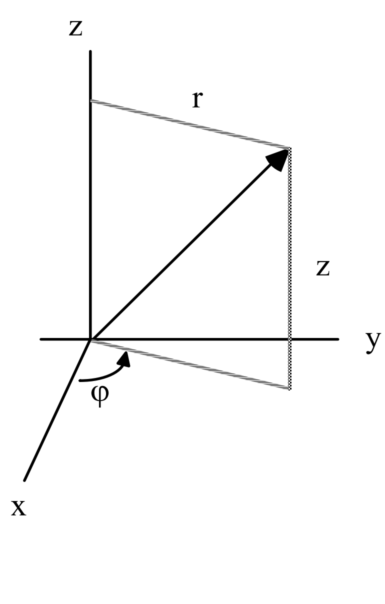
Coordinates (r, φ, z) Unit vectors $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$

Gradient
$$\nabla f = \vec{i}_1 \frac{\partial f}{\partial r} + \vec{i}_2 \frac{1}{r}\frac{\partial f}{\partial \varphi} + \vec{i}_3 \frac{\partial f}{\partial z}$$

Curl
$$\nabla \times \vec{A} = \vec{i}_1\left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \vec{i}_2\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \vec{i}_3\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial \varphi}\right)$$

Divergence
$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r\,A_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

Laplacian
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$



VECTOR OPERATIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

SPHERICAL COORDINATES

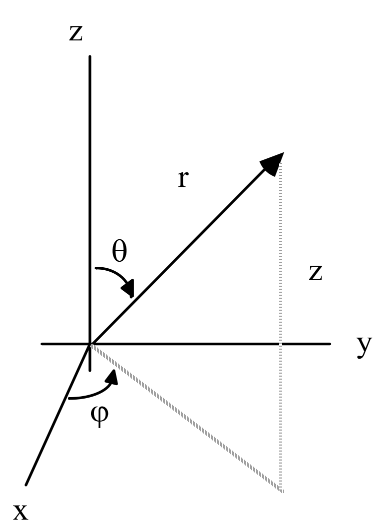
Coordinates (r, θ, φ) Unit vectors $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$

Gradient $$\nabla f = \vec{i}_1 \frac{\partial f}{\partial r} + \vec{i}_2 \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{i}_3 \frac{1}{r \sin\theta}\frac{\partial f}{\partial \varphi}$$

Curl $$\nabla \times \vec{A} = \vec{i}_1 \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right) + \vec{i}_2\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r\, A_\varphi)\right)$$
$$+ \vec{i}_3\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$$

Divergence $$\nabla \bullet \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

Laplacian $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$
$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$



$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f \qquad \vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f) \qquad \vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}f)$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \qquad \vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

$$P_o(\cos\theta) = 1;\; P_1(\cos\theta) = \cos\theta;\; P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1);\; P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \tfrac{2}{2l+1}\delta_{l,l'} \quad ; \quad \int_0^{a}\sin(n\pi x/a)\sin(n'\pi x/a)dx = \tfrac{a}{2}\delta_{n,n'}$$

$$\cos 2\theta = (2\cos^2\theta - 1) \quad ; \quad \sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin(3\theta) \quad ; \quad \int_0^{\infty} r^n e^{-\alpha r}dr = \tfrac{n!}{\alpha^{n+1}}$$

$$\int_0^{2\pi}\int_0^{\pi} Y^{*m}_{l+1}(\theta,\phi)Y_1^0(\theta,\phi)Y_l^m(\theta,\phi)\sin\theta d\theta d\phi = \sqrt{\frac{3}{4\pi}}\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}$$

$$\int_0^{2\pi}\int_0^{\pi} Y^{*m+1}_{l+1}(\theta,\phi)Y_1^1(\theta,\phi)Y_l^m(\theta,\phi)\sin\theta d\theta d\phi = \sqrt{\frac{3}{8\pi}}\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}$$

$$\int_0^{2\pi}\int_0^{\pi} Y^{*m+1}_{l-1}(\theta,\phi)Y_1^1(\theta,\phi)Y_l^m(\theta,\phi)\sin\theta d\theta d\phi = -\sqrt{\frac{3}{8\pi}}\sqrt{\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}}$$

$$Y_l^{-m}(\theta,\phi) = (-1)^m Y^{*m}_l(\theta,\phi)$$

$$\int\frac{dx}{(x^2+a^2)^{\frac{1}{2}}} = \ln[x + (x^2+a^2)^{\frac{1}{2}}] \qquad \int(x^2+a^2)^{\frac{1}{2}}dx = \tfrac{x}{2}(x^2+a^2)^{\frac{1}{2}} + \tfrac{a^2}{2}\ln[x+(x^2+a^2)^{\frac{1}{2}})$$

$$\int\frac{xdx}{(x^2+a^2)^{\frac{1}{2}}} = (x^2+a^2)^{\frac{1}{2}} \qquad \int x^2(x^2+a^2)^{\frac{1}{2}}dx = \tfrac{x}{8}(x^2+a^2)^{\frac{1}{2}}(2x^2+a^2) - \tfrac{a^4}{8}\ln[x+(x^2+a^2)^{\frac{1}{2}})$$

$$\int\frac{x^2dx}{(x^2+a^2)^{\frac{1}{2}}} = \tfrac{x}{2}(x^2+a^2)^{\frac{1}{2}} - \tfrac{a^2}{2}\ln[x+(x^2+a^2)^{\frac{1}{2}}] \qquad \int\frac{x^3dx}{(x^2+a^2)^{\frac{1}{2}}} = \tfrac{1}{3}(x^2+a^2)^{\frac{1}{2}} - a^2(x^2+a^2)^{\frac{1}{2}}$$

$$\int\frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2(x^2+a^2)^{\frac{1}{2}}} \qquad \int\frac{xdx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{-1}{(x^2+a^2)^{\frac{1}{2}}}$$

$$\int\frac{x^2dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{-x}{(x^2+a^2)^{\frac{1}{2}}} + \ln[x+(x^2+a^2)^{\frac{1}{2}}] \qquad \int\frac{x^3dx}{(x^2+a^2)^{\frac{3}{2}}} = (x^2+a^2)^{\frac{1}{2}} + \frac{a^2}{(x^2+a^2)^{\frac{1}{2}}}$$

Binomial expansion: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ where $x^2 < 1$

Some useful constants: $e = 1.6 \times 10^{-19}$ C; $\varepsilon_o = 8.85 \times 10^{-12}$ C^2/Nm