EM-A, Solutions

When the charge $q$ is at $x$, its image $-q$ is at $-x$, so the force between them is

$$F = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{4x^2} = -\frac{dU}{dx}. $$

Thus $U = -\frac{q^2}{16\pi\varepsilon_0x}$. By energy conservation

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{q^2}{16\pi\varepsilon_0x} = -\frac{q^2}{16\pi\varepsilon_0D}$$

Thus

$$\frac{dx}{dt} = \sqrt{\frac{q^2}{8\pi\varepsilon_0m} \left( \frac{D - x}{xD} \right)}.$$ 

$$t = \sqrt{\frac{8\pi\varepsilon_0mD}{q^2}} \int_0^{D} \frac{\sqrt{x}dx}{\sqrt{D - x}}.$$ 

To do this integral, write $x = D \sin^2 y$, getting $t = \frac{q}{q} \sqrt{2\pi\varepsilon_0mD^3}$. 

EM-B Solutions

\[ \vec{E}_{\text{whole}}(\vec{r}) = \vec{E}_{\text{cavity}}(\vec{r}) + \vec{E}_{\text{part}}(\vec{r}) \]

Gauss’s law for the whole:

\[ \oint \vec{E}_{\text{whole}} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \frac{4\pi}{3} r^3 \rho \]

\[ \Rightarrow \vec{E}_{\text{whole}}(\vec{r}) 4\pi r^2 = \frac{4\pi}{3} \left( \frac{\rho}{\varepsilon_0} \right) \]

\[ \Rightarrow \vec{E}_{\text{whole}}(\vec{r}) = \frac{\vec{r}}{3} \left( \frac{\rho}{\varepsilon_0} \right) \]

Similarly:

\[ \Rightarrow \vec{E}_{\text{part}}(\vec{r}) = \frac{\vec{r}'}{3} \left( \frac{\rho}{\varepsilon_0} \right) \]

\[ \Rightarrow \vec{E}_{\text{part}}(\vec{r}) = \frac{\vec{r} - \vec{a}}{3} \left( \frac{\rho}{\varepsilon_0} \right) \]

So:

\[ \vec{E}_{\text{cavity}}(\vec{r}) = \vec{E}_{\text{whole}}(\vec{r}) - \vec{E}_{\text{part}}(\vec{r}) \]

\[ = \frac{\vec{r}}{3} \left( \frac{\rho}{\varepsilon_0} \right) - \frac{\vec{r} - \vec{a}}{3} \left( \frac{\rho}{\varepsilon_0} \right) \]

\[ = \frac{\vec{a}}{3} \left( \frac{\rho}{\varepsilon_0} \right) \]

\[ r' = r - a \]

Whole

Cavity

Part
EM C-1, Solutions

In units with $c = 1$,

$P_1 = (\gamma m, 0, 0, \gamma \beta m)$

$P_2 = (m, 0, 0, 0)$.

Since $P_i^2 = m^2$, energy and momentum conservation gives $P_3 \cdot P_4 = P_1 \cdot P_2 = \gamma m^2$.

If both final particles have the same energy $E$, the magnitudes of their momenta must also be equal. Thus

$\gamma m^2 = (P_3 \cdot P_4) = E^2 - (E^2 - m^2) \cos \vartheta$.

where by energy conservation

$2E = \gamma m + m$,

giving

$\cos \vartheta = \frac{2\gamma - 1}{\gamma + 3}$
Solution EM_C2

(a) An electromagnetic plane wave is given by
\[ E = E_0 e^{i(kr - \omega t)} \quad H = H_0 e^{i(kr - \omega t)} \quad B = \mu H \quad D = \varepsilon E, \]
where taking the real part is implied. The divergences of \( D \) and \( B \) are zero provided that \( \mathbf{k} \perp \mathbf{E}_0 \) and \( \mathbf{k} \perp \mathbf{H}_0 \), satisfying two of the Maxwell equations. The other two, involving the curl of \( \mathbf{E} \) and \( \mathbf{H} \), in the absence of free sources read \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) and \( \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \), which upon substitution of the wave become
\[ \mathbf{k} \times \mathbf{E}_0 = \omega \mu \mathbf{H}_0 \]  
(1)
and
\[ \mathbf{k} \times \mathbf{H}_0 = -\omega \varepsilon \mathbf{E}_0. \]  
(2)
It is equations (1) and (2) we now seek to satisfy. For \( \varepsilon > 0 \) and \( \mu > 0 \) we find, by combining the two equations: \( \mathbf{k} \times \mathbf{k} \times \mathbf{E}_0 = -\omega^2 \varepsilon \mu \mathbf{E}_0 \), which is satisfied for \( \mathbf{k} \perp \mathbf{E}_0 \) and \( k = \omega \sqrt{\varepsilon \mu} \). Choosing \( \mathbf{E}_0 \) along the \( x \)-axis and \( \mathbf{k} \) along the \( z \)-axis, the solution (in Cartesian column vector notation) therefore is
\[ \mathbf{k} = \sqrt{\varepsilon \mu} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad \mathbf{E}_0 = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{H}_0 = \sqrt{\frac{\varepsilon}{\mu}} \begin{pmatrix} 0 \\ E_0 \end{pmatrix}. \]
The vectors \( \mathbf{k}, \mathbf{E}, \) and \( \mathbf{H} \) form a right-handed triplet.

The Poynting vector is given by \( \mathbf{S} = \text{Re}(\mathbf{E}) \times \text{Re}(\mathbf{H}) = \frac{\varepsilon}{\mu} \begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix} \cos^2(kz - \omega t) \). Upon time-averaging, we get the average energy flux \( \langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix} \). Note that the Poynting vector points (poynts?) in the same direction as \( \mathbf{k} \). There is no energy dissipation.
(b) \( \varepsilon < 0 \) and \( \mu > 0 \):  
Keeping \( \mathbf{E}_0 \) real, we find that to satisfy equations (1) and (2), \( \mathbf{k} \) and \( \mathbf{H}_0 \) must be imaginary:

\[
\mathbf{k} = -i \sqrt{\varepsilon |\mu|} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad \mathbf{E}_0 = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{H}_0 = -i \sqrt{\frac{\varepsilon}{\mu}} \begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}.
\]

Due to the imaginary \( \mathbf{k} \), the wave is nonoscillatory in space but decays exponentially. It does not propagate. Instead, it is an evanescent wave.

The Poynting vector is \( \mathbf{S} = \text{Re}(\mathbf{E}) \times \text{Re}(\mathbf{H}) = \sqrt{\frac{\varepsilon}{\mu}} \begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix} \cos(kz - \omega t) \sin(kz - \omega t) \), and its time average is zero \( \langle \mathbf{S} \rangle = 0 \). The evanescent wave does not transport energy. There is no energy dissipation. The medium does not support propagating waves.

\( \varepsilon > 0 \) and \( \mu < 0 \):  
Keeping \( \mathbf{E}_0 \) real, we find that to satisfy equations (1) and (2), \( \mathbf{k} \) and \( \mathbf{H}_0 \) must be imaginary:

\[
\mathbf{k} = i \sqrt{\varepsilon |\mu|} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad \mathbf{E}_0 = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{H}_0 = -i \sqrt{\frac{\varepsilon}{\mu}} \begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}.
\]

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(c) $\varepsilon < 0$ and $\mu < 0$:

In this case $E_0$, $k$ and $H_0$ are all real again:

$$k = -\sqrt{\varepsilon \mu} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \quad E_0 = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix}, \quad H_0 = \sqrt{\frac{\varepsilon}{\mu}} E_0 \begin{pmatrix} 0 \\ E_0 \end{pmatrix}.$$

These vectors form a left-handed triplet. Due to the real $k$, the wave oscillates in space and propagates, just like in the ordinary case (a). The difference is that $k$ is reversed.

The Poynting vector is $S = \text{Re}(E) \times \text{Re}(H) = \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \cos^2(kz - \omega t)$, and its time average is

$$\langle S \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} E_0^2,$$

just like in case (a). However, in this case it points in the opposite direction of $k$: The phase fronts move in one direction, while the energy flows in the other. There is no energy dissipation.
Solution EM_D1

(a)

Inductance: \( L = \mu_0 r \left( \ln \frac{8r}{a} - 2 \right) = \beta \mu_0 d \), with geometry factor \( \beta = 2(\ln 32 - 2) \approx 2.931 \)

Capacitor plate area: \( A = \pi \left( \frac{d}{2} \right)^2 \)

Capacitance: \( C = \gamma \varepsilon_0 A = \frac{\gamma \pi}{4} \varepsilon_0 d \)

Resonance frequency: \( \omega_0 = \frac{1}{\sqrt{LC}} = \frac{2c}{d \pi \beta \gamma} \Rightarrow f = \frac{\omega_0}{2\pi} \approx 0.0692 \frac{c}{d} \) 

(with speed of light \( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s} \))

The resonance frequency thus increases like \( 1/d \) as \( d \) is reduced.

(b)

Consider a uniform displacement of the electrons along the wire by a small distance \( x \). This will charge the capacitor with a charge \( Q = An e x \), where \( e \) is the charge of the electron. The capacitor voltage is \( V = Q / C \), and inside the conductor we have an electric restoring field \( E \) along the wire which integrates to \( V \) as we go from one capacitor plate around the ring to the other: \( V = \int Edl \). To obtain the total restoring force on all the electrons (assumed rigid) we perform a volume integral over the ring, \( F = \int neEd\Omega = Ane \int Edl = AneV = \frac{An^2 e^2 xd}{\gamma \varepsilon_0} \).

Since the restoring force is proportional to the displacement \( x \), we may define a spring constant \( k = \frac{F}{x} \) for this degree of freedom, and obtain the resonance frequency \( f = \frac{1}{2\pi} \sqrt{k} \), using the oscillating mass of electrons \( m = nA m_e \). \( l \) is the length of the wire around the ring: \( l = 2\pi r - d = d(4\pi - 1) \), where we have subtracted the gap width \( d \) from the circumference.

Putting it all together, we have \( f = \frac{\alpha}{2\pi} \omega_p \), where \( \omega_p \) is the bulk plasma frequency

\[ \omega_p = \sqrt{\frac{ne^2}{m \varepsilon_0}} \approx 1.783 \times 10^{16} \text{ rad/s}, \quad \text{and } \alpha \text{ is a geometry factor given by} \]
The resonance frequency thus becomes \( f \approx 5.50 \times 10^{14} \) Hz, independent of the length scale \( d \)! (This frequency corresponds to visible light in the yellow-green part of the spectrum. If \( d \) is small enough, we thus expect our split-ring resonator to efficiently absorb light of this wavelength.)

(e)

We set up an effective Lagrangian for the oscillation:

\[
\Lambda = KE - PE, \quad \text{with} \quad PE = \frac{1}{2} \frac{Q^2}{C}
\]

and a kinetic term containing both the actual kinetic energy and the inductive energy: \( KE = \frac{1}{2} mv^2 + \frac{1}{2} LI^2 \), where \( v \) is the drift velocity of the electrons and \( I \) is the current. The latter quantities are related via \( I = Anev \), so the Lagrangian becomes \( \Lambda = \frac{1}{2} L' I^2 - \frac{1}{2} \frac{Q^2}{C} \), with the “effective” inductance \( L' = L + \frac{lm}{Ane^2} \). Since \( I = \dot{Q} \), the equation of motion is \( \frac{d}{dt} \frac{\partial \Lambda}{\partial I} = \frac{\partial \Lambda}{\partial Q} \), which becomes \( \dot{Q} = -\frac{1}{L' C} Q \) and is solved by an oscillation at angular frequency \( \omega = \frac{1}{\sqrt{L'C}} = \frac{1}{\sqrt{LC + \frac{1}{\alpha^2 \omega_p^2}}} \). To determine the crossover length scale, we set \( \frac{\omega_d}{\omega_p} = \frac{1}{\alpha^2 \omega_p^2} \) and solve for \( d \), obtaining

\[
d_0 = \frac{2}{\alpha \sqrt{\pi \beta \gamma}} \frac{c}{\omega_p} \approx 2.24 \frac{c}{\omega_p} \approx 38 \text{ nm.}
\]

For \( d >> 38 \text{ nm} \), the resonance frequency approaches \( \omega_0 = \frac{1}{\sqrt{LC}} \propto d^{-1} \), while for \( d << 38 \text{ nm} \), the resonance frequency becomes independent of \( d \): \( \omega = \alpha \omega_p \).
EM-D2, Solution

(a) We use Gauss’ law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

Using a cylindrical Gaussian surface of height $\ell$, radius $r$

$$2\pi r\ell E_r = \frac{\ell \lambda}{\varepsilon}$$

Therefore

$$E_r = \frac{1}{2\pi \varepsilon} \frac{\lambda}{r}$$

Potential of the inner surface w.r.t. the outer surface is

$$\Delta V = - \int_{r=b}^{a} \mathbf{E} \cdot d\mathbf{r} = - \int_{b}^{a} \frac{1}{2\pi \varepsilon} \frac{\lambda}{r} dr$$

$$= \frac{\lambda}{2\pi \varepsilon} \ln(b/a)$$

$$Q = \Delta VC\ell$$

$$\rightarrow C = \frac{\lambda}{\Delta V} = \frac{2\pi}{\ln(b/a)}$$
(b) Magnetic

Energy stored in the coaxial cable per unit length is

\[ U = \frac{1}{2} \frac{1}{\mu_0} \int_{\text{unit length}} B^2 d^3r \]

\[ = \frac{1}{2} LI^2 \]  

\[ \nabla \times B = \mu_0 J \Rightarrow \oint B \cdot dl = \mu_0 I \]

\[ \Rightarrow 2\pi r B_\phi = \mu_0 I \Rightarrow B_\phi = \frac{\mu_0 I}{2\pi r}, a < r < b \]

[symmetry implies \( B = B_\phi \hat{\phi} \)]

\[ \Rightarrow U = \frac{1}{2\mu_0} \left(\frac{\mu_0}{2\pi}\right)^2 I^2 \int_{r=a}^{b} \frac{1}{r^2} d^3\gamma \]

\[ = \frac{\mu_0}{8\pi^2} I^2 \int_{r=a}^{b} \frac{1}{r^2} 2\pi r dr \]

\[ = \frac{\mu_0}{4\pi} I^2 \ln(b/a) \]

\[ = \frac{1}{2} LI^2 \]  

Therefore: \( L = \frac{\mu_0}{2\pi} \ln(b/a) \)
\[
V(x) - Ldx \frac{\partial I}{\partial t} - IRdx = V(x + dx) \quad (6)
\]

\[
I(x) - cdx \frac{\partial V}{\partial t} = I(x + dx)
\]

\[
\Rightarrow dV(x) = -Ldx \frac{\partial I}{\partial t} - IRdx
\]

\[
\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} - IR
\]

\[
dI(x) = -cdx \frac{\partial v}{\partial t} \quad (7)
\]

\[
\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}
\]

\[
\Rightarrow \frac{\partial^2 I}{\partial x^2} = -C \frac{\partial^2 V}{\partial x \partial t} = +c \left[ L \frac{\partial^2 I}{\partial t^2} + R \frac{\partial I}{\partial t} \right] \quad (8)
\]

\[
\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2} + RC \frac{\partial I}{\partial t}
\]

Similarly

\[
\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} + RC \frac{\partial V}{\partial t} \quad (9)
\]
(d) \[
\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2}
\] (10)

With \( I = I_o e^{i(kx - \omega t)} \)

\[ \Rightarrow -k^2 = -LC\omega^2 \rightarrow k = \pm \sqrt{LC}\omega \]

since the signal propagates along +x, \( k = \sqrt{LC}\omega \)

Therefore:

\[ I = I_o \cos \left( \sqrt{LC}\omega x - \omega t + \phi \right) \] (11)

where \( I_o \) and \( \phi \) need to be determined.

part (c) gives

\[
\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}
\] (12)

\[
\frac{\partial I}{\partial x} \bigg|_{x=0} = -I_o \sqrt{LC}\omega \sin(-\omega t + \phi)
\]

\[
\frac{\partial V}{\partial t} \bigg|_{x=0} = -V_o \omega \sin(\omega t)
\] (13)

since \( \frac{\partial I}{\partial x} \bigg|_{x=0} = -C \frac{\partial V}{\partial t} \bigg|_{x=0} \) at all times,

and \( I_o \sqrt{LC}\omega = CV_o \omega \Rightarrow I_o = \sqrt{\frac{C}{L}}V_o \)

Thus

\[ I(x, t) = \sqrt{\frac{C}{L}} V_o \cos(\sqrt{LC}\omega x - \omega t) \] (14)

\[
Z_e = \frac{V(x, t)}{I(x, t)} = \sqrt{\frac{L}{C}}
\] (15)