Physics Qualifying Exam, August 2008
Electricity and Magnetism

EM-A, Solutions

When the charge q is at x, its image $-q$ is at $-x$, so the force between them is

$$F = -\frac{1}{4\pi\varepsilon_O}\frac{q^2}{4x^2} = -\frac{dU}{dx}.$$

Thus U $= -\frac{q^2}{16\pi\varepsilon_o x}$. By energy conservation

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{q^2}{16\pi\varepsilon_o x} = -\frac{q^2}{16\pi\varepsilon_o D}$$

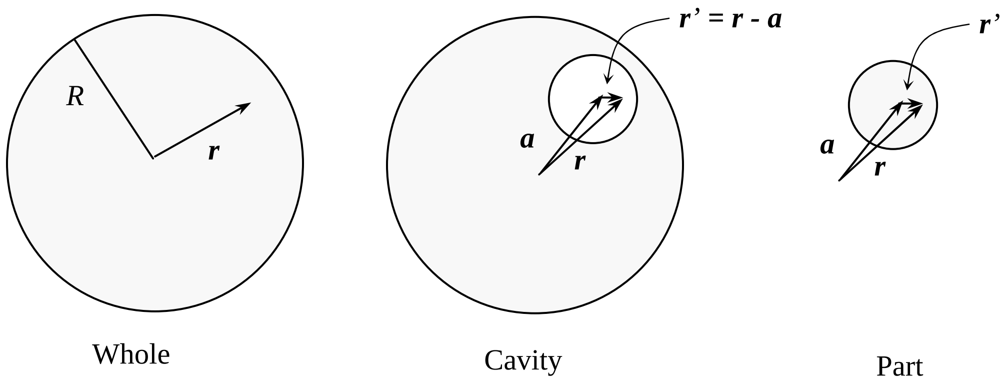
Thus

$$\frac{dx}{dt} = \sqrt{\frac{q^2}{8\pi\varepsilon_o m}\left(\frac{D-x}{xD}\right)}.$$

$$t = \sqrt{\frac{8\pi\varepsilon_o mD}{q^2}}\int_o^D \frac{\sqrt{x}dx}{\sqrt{D-x}}.$$

To do this integral, write $x = D\sin^2 y$, getting $t = \frac{\pi}{q}\sqrt{2\pi\varepsilon_0 mD^3}$.

EM-B Solutions



$$\vec{E}_{whole}(\vec{r}) = \vec{E}_{cavity}(\vec{r}) + \vec{E}_{part}(\vec{r})$$

Gauss's law for the whole:

$$\oint \vec{E}_{whole} \cdot d\vec{A} = \frac{1}{\varepsilon_o}\frac{4\pi}{3} r^3 \rho$$

$$\Rightarrow \vec{E}_{whole}(\vec{r}) 4\pi r^2 = \frac{4\pi r^3}{3}\left(\frac{\rho}{\varepsilon_o}\right)$$

$$\Rightarrow \vec{E}_{whole}(\vec{r}) = \frac{\vec{r}}{3}\left(\frac{\rho}{\varepsilon_o}\right)$$

Similarly:

$$\Rightarrow \vec{E}_{part}(\vec{r}) = \frac{\vec{r}\,'}{3}\left(\frac{\rho}{\varepsilon_o}\right)$$

$$\Rightarrow \vec{E}_{part}(\vec{r}) = \frac{\vec{r} - \vec{a}}{3}\left(\frac{\rho}{\varepsilon_o}\right)$$

So:

$$\vec{E}_{cavity}(\vec{r}) = \vec{E}_{whole}(\vec{r}) - \vec{E}_{part}(\vec{r})$$

$$= \frac{\vec{r}}{3}\left(\frac{\rho}{\varepsilon_o}\right) - \frac{\vec{r} - \vec{a}}{3}\left(\frac{\rho}{\varepsilon_o}\right)$$

$$= \frac{\vec{a}}{3}\left(\frac{\rho}{\varepsilon_o}\right)$$

EM C-1, Solutions

In units with $c = 1$,

$P_1 = (\gamma m, 0, 0, \gamma\beta m)$

$P_2 = (m, 0, 0, 0)$.

Since $P_i^2 = m^2$, energy and momentum conservation gives $P_3 \cdot P_4 = P_1 \cdot P_2 = \gamma m^2$.

If both final particles have the same energy E, the magnitudes of their momenta must also be equal. Thus

$\gamma m^2 = (P_3 \cdot P_4) = E^2 - (E^2 - m^2)\cos\vartheta$.

where by energy conservation

$2E = \gamma m + m$,

giving

$\cos\vartheta = \frac{\gamma - 1}{\gamma + 3}$

Solution EM_C2

(a) An electromagnetic plane wave is given by

$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}$ $\quad$ $\mathbf{H} = \mathbf{H}_0 e^{i(\mathbf{kr}-\omega t)}$ $\quad$ $\mathbf{B} = \mu\mathbf{H}$ $\quad$ $\mathbf{D} = \varepsilon\mathbf{E}$, where taking the real part is implied. The divergences of **D** and **B** are zero provided that $\mathbf{k} \perp \mathbf{E}_0$ and $\mathbf{k} \perp \mathbf{H}_0$, satisfying two of the Maxwell equations. The other two, involving the curl of **E** and **H**, in the absence of free sources read $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$, which upon substitution of the wave become

$$\mathbf{k} \times \mathbf{E}_0 = \omega\mu\mathbf{H}_0 \tag{1}$$

and

$$\mathbf{k} \times \mathbf{H}_0 = -\omega\varepsilon\mathbf{E}_0 . \tag{2}$$

It is equations (1) and (2) we now seek to satisfy. For $\varepsilon > 0$ and $\mu > 0$ we find, by combining the two equations: $\mathbf{k} \times \mathbf{k} \times \mathbf{E}_0 = -\omega^2 \varepsilon\mu\mathbf{E}_0$, which is satisfied for $\mathbf{k} \perp \mathbf{E}_0$ and $k = \omega\sqrt{\varepsilon\mu}$. Choosing $\mathbf{E}_0$ along the x-axis and **k** along the z-axis, the solution (in Cartesian column vector notation) therefore is

$$\mathbf{k} = \sqrt{\varepsilon\mu}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \qquad \mathbf{E}_0 = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{H}_0 = \sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}.$$

The vectors **k**, **E**, and **H** form a right-handed triplet.

The Poynting vector is given by $\mathbf{S} = \mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{H}) = \sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix}\cos^2(kz - \omega t)$. Upon time-averaging, we get the average energy flux $\langle \mathbf{S} \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix}$. Note that the Poynting vector points (poynts?) in the same direction as **k**. There is no energy dissipation.

(b) $\varepsilon < 0$ and $\mu > 0$:

Keeping $\mathbf{E}_0$ real, we find that to satisfy equations (1) and (2), **k** and $\mathbf{H}_0$ must be imaginary:

$$\mathbf{k} = -i\sqrt{|\varepsilon|\mu}\begin{pmatrix}0\\0\\\omega\end{pmatrix} \qquad \mathbf{E}_0 = \begin{pmatrix}E_0\\0\\0\end{pmatrix} \qquad \mathbf{H}_0 = -i\sqrt{\frac{|\varepsilon|}{\mu}}\begin{pmatrix}0\\E_0\\0\end{pmatrix}.$$

Due to the imaginary **k**, the wave is nonoscillatory in space but decays exponentially. It does not propagate. Instead, it is an evanescent wave.

The Poynting vector is $\mathbf{S} = \mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{H}) = \sqrt{\frac{|\varepsilon|}{\mu}}\begin{pmatrix}0\\0\\E_0^2\end{pmatrix}\cos(kz-\omega t)\sin(kz-\omega t)$, and its time average is zero $\langle \mathbf{S}\rangle = 0$. The evanescent wave does not transport energy. There is no energy dissipation. The medium does not support propagating waves.

$\varepsilon > 0$ and $\mu < 0$:

Keeping $\mathbf{E}_0$ real, we find that to satisfy equations (1) and (2), **k** and $\mathbf{H}_0$ must be imaginary:

$$\mathbf{k} = i\sqrt{\varepsilon|\mu|}\begin{pmatrix}0\\0\\\omega\end{pmatrix} \qquad \mathbf{E}_0 = \begin{pmatrix}E_0\\0\\0\end{pmatrix} \qquad \mathbf{H}_0 = -i\sqrt{\frac{\varepsilon}{|\mu|}}\begin{pmatrix}0\\E_0\\0\end{pmatrix}.$$

Due to the imaginary **k**, the wave is nonoscillatory in space but decays exponentially. It does not propagate. Instead, it is an evanescent wave.

The Poynting vector is $\mathbf{S} = \mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{H}) = \sqrt{\frac{\varepsilon}{|\mu|}}\begin{pmatrix}0\\0\\E_0^2\end{pmatrix}\cos(kz-\omega t)\sin(kz-\omega t)$, and its time average is zero $\langle \mathbf{S}\rangle = 0$. The evanescent wave does not transport energy. There is no energy dissipation. The medium does not support propagating waves.

(c) $\varepsilon < 0$ and $\mu < 0$:

In this case $\mathbf{E}_0$, $\mathbf{k}$ and $\mathbf{H}_0$ are all real again:

$$\mathbf{k} = -\sqrt{\varepsilon\mu}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \qquad \mathbf{E}_0 = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{H}_0 = \sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}.$$

These vectors form a left-handed triplet. Due to the real **k**, the wave oscillates in space and propagates, just like in the ordinary case (a). The difference is that **k** is reversed.

The Poynting vector is $\mathbf{S} = \mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{H}) = \sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix}\cos^2(kz - \omega t)$, and its time average is

$\langle \mathbf{S} \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\begin{pmatrix} 0 \\ 0 \\ E_0^2 \end{pmatrix}$, just like in case (a). However, in this case it poynts in the opposite direction of **k**: The phase fronts move in one direction, while the energy flows in the other. There is no energy dissipation.

Solution EM_D1

(a)

Inductance: $L = \mu_0 r\left(\ln\frac{8r}{a} - 2\right) = \beta\mu_0 d$, with geometry factor $\beta = 2(\ln 32 - 2) \approx 2.931$

Capacitor plate area: $A = \pi\left(\frac{d}{2}\right)^2$

Capacitance: $C = \gamma\varepsilon_0 \frac{A}{d} = \frac{\gamma\pi}{4}\varepsilon_0 d$

Resonance frequency: $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{2c}{d\sqrt{\pi\beta\gamma}}$ $\qquad f = \frac{\omega_0}{2\pi} \approx 0.0692\frac{c}{d}$

(with speed of light $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times 10^8$ m/s)

The resonance frequency thus increases like $1/d$ as d is reduced.

(b)

Consider a uniform displacement of the electrons along the wire by a small distance x. This will charge the capacitor with a charge $Q = Anex$, where e is the charge of the electron. The capacitor voltage is $V = Q/C$, and inside the conductor we have an electric restoring field E along the wire which integrates to V as we go from one capacitor plate around the ring to the other: $V = \int Edl$. To obtain the total restoring force on all the electrons (assumed rigid) we perform a volume integral over the ring, $F = \int neEd\Omega = Ane\int Edl = AneV = \frac{An^2e^2xd}{\gamma\varepsilon_0}$.

Since the restoring force is proportional to the displacement x, we may define a spring constant $k = \frac{F}{x}$ for this degree of freedom, and obtain the resonance frequency $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, using the oscillating mass of electrons $m = nAlm_e$. l is the length of the wire around the ring: $l = 2\pi r - d = d(4\pi - 1)$, where we have subtracted the gap width d from the circumference.

Putting it all together, we have $f = \frac{\alpha}{2\pi}\omega_p$, where ω_p is the bulk plasma frequency $\omega_p = \sqrt{\frac{ne^2}{m\varepsilon_0}} \approx 1.783\times 10^{16}$ rad/s, and α is a geometry factor given by

$\alpha = \frac{1}{\sqrt{\gamma(4\pi - 1)}} \approx 0.194$. The resonance frequency thus becomes $f \approx 5.50 \times 10^{14}$ Hz, independent of the length scale *d*! (This frequency corresponds to visible light in the yellow-green part of the spectrum. If *d* is small enough, we thus expect our split-ring resonator to efficiently absorb light of this wavelength.)

(c)

We set up an effective Lagrangian for the oscillation:

$\Lambda = KE - PE$, with $PE = \frac{1}{2}\frac{Q^2}{C}$ and a kinetic term containing both the actual kinetic energy and the inductive energy: $KE = \frac{1}{2}mv^2 + \frac{1}{2}LI^2$, where *v* is the drift velocity of the electrons and *I* is the current. The latter quantities are related via $I = Anev$, so the Lagrangian becomes $\Lambda = \frac{1}{2}L'I^2 - \frac{1}{2}\frac{Q^2}{C}$, with the "effective" inductance $L' = L + \frac{lm_e}{Ane^2}$. Since $I = \dot{Q}$, the equation of motion is $\frac{d}{dt}\frac{\partial\Lambda}{\partial I} = \frac{\partial\Lambda}{\partial Q}$, which becomes $\ddot{Q} = -\frac{1}{L'C}Q$ and is solved by an oscillation at angular frequency $\omega = \frac{1}{\sqrt{L'C}} = \frac{1}{\sqrt{LC + \frac{1}{\alpha^2\omega_p{}^2}}} = \frac{1}{\sqrt{\frac{1}{\omega_0{}^2} + \frac{1}{\alpha^2\omega_p{}^2}}}$. To determine the crossover length scale, we set $\frac{1}{\omega_0{}^2} = \frac{1}{\alpha^2\omega_p{}^2}$ and solve for *d,* obtaining

$d_0 = \frac{2}{\alpha\sqrt{\pi\beta\gamma}}\frac{c}{\omega_p} \approx 2.24\frac{c}{\omega_p} \approx 38$ nm. For $d >> 38$ nm the resonance frequency approaches $\omega_0 = \frac{1}{\sqrt{LC}} \propto d^{-1}$, while for $d << 38$ nm the resonance frequency becomes independent of *d:* $\omega = \alpha\omega_p$.

EM-D2, Solution

(a)We use Gauss' law:

$$\nabla \cdot E = \frac{\rho}{\varepsilon}$$

Using a cylindrical Gaussian surface of height ℓ, radius r

$$2\pi r \ell E_r = \frac{\ell \lambda}{\varepsilon}$$

Therefore

$$E_r = \frac{1}{2\pi\varepsilon}\frac{\lambda}{r}$$

Potential of the inner surface w.r.t. the outer surface is

$$\Delta V = -\int_{r=b}^{a} \mathbf{E} \cdot d\mathbf{r} = -\int_{b}^{a} \frac{1}{2\pi\varepsilon}\frac{\lambda}{r} dr$$

$$= \frac{\lambda}{2\pi\varepsilon} \ln(b/a)$$

$$Q = \Delta V C \ell$$

$$\rightarrow C = \frac{\lambda}{\Delta V} = \frac{2\pi}{\ln(b/a)}$$

(b) Magnetic
Energy stored in the coaxial cable per unit length is

$$\begin{aligned} U &= \frac{1}{2}\frac{1}{\mu_o}\int_{unitlength} B^2 d^3r \qquad (1) \\ &= \frac{1}{2}LI^2 \\ \nabla \times \mathbf{B} = \mu_o \mathbf{J} &\Rightarrow \oint \mathbf{B}\cdot d\mathbf{l} = \mu_o I \\ \Rightarrow 2\pi r B_\phi = \mu_o I \Rightarrow B_\phi &= \frac{\mu_o}{2\pi}\frac{I}{r}, a < r < b \end{aligned}$$

[symmetry implies $\mathbf{B} = B_\phi \hat{\phi}$]

$$\begin{aligned} \Rightarrow U &= \frac{1}{2\mu_o}\left(\frac{\mu_O}{2\pi}\right)^2 I^2 \int_{\gamma=a}^{b} \frac{1}{r^2} d^3\gamma \qquad (2) \\ &= \frac{\mu_o}{8\pi^2} I^2 \int_{r=a}^{b} \frac{1}{r^2} 2\pi r dr \qquad (3) \\ &= \frac{\mu_o}{4\pi} I^2 \ln(b/a) \qquad (4) \\ &= \frac{1}{2}LI^2 \qquad (5) \end{aligned}$$

Therefore: $L = \frac{\mu_o}{2\pi}\ln(b/a)$

(c)

$$V(x) - Ldx\frac{\partial I}{\partial t} - IRdx = V(x+dx) \tag{6}$$

$$I(x) - cdx\frac{\partial V}{\partial t} = I(x+dx)$$

$$\Rightarrow dV(x) = -Ldx\frac{\partial I}{\partial t} - IRdx$$

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - IR$$

$$dI(x) = -cdx\frac{\partial v}{\partial t} \tag{7}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{\partial^2 I}{\partial x^2} = -C\frac{\partial^2 V}{\partial x \partial t} = +c\left[L\frac{\partial^2 I}{\partial t^2} + R\frac{\partial I}{\partial t}\right] \tag{8}$$

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + RC\frac{\partial I}{\partial t}$$

Similarly

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t} \tag{9}$$

(d)

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \tag{10}$$

With $I = I_o\ \mathrm{e}^{i(kx-wt)}$

$$\Rightarrow -k^2 = -LC\omega^2 \rightarrow k = \pm\sqrt{LC}\omega$$

since the signal propagates along $+x$, $k = \sqrt{LC}\omega$
Therefore:

$$I = I_o cos\left(\sqrt{LC}\omega x - \omega t + \phi\right) \tag{11}$$

where I_o, and ϕ need to be determined.
part (c) gives

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \tag{12}$$

$$\begin{aligned}\frac{\partial I}{\partial x}|_{x=o} &= -I_o\sqrt{LC}\omega sin\left(-\omega t + \phi\right)\\ \frac{\partial V}{\partial t}|_{x=o} &= -V_o\omega sin(\omega t)\end{aligned} \tag{13}$$

since $\frac{\partial I}{\partial x}\Big|_{x=o} = -C\frac{\partial V}{\partial t}\Big|_{x=o}$ at all times,

and $I_o\sqrt{LC}\omega = CV_o\omega \Rightarrow I_o = \sqrt{\frac{C}{L}}V_o$
Thus

$$I(x,t) = \sqrt{\frac{C}{L}}V_o cos(\sqrt{LC}\omega x - \omega t) \tag{14}$$

$$Z_c \equiv \frac{V(x,t)}{I(x.t)} = \sqrt{\frac{L}{C}} \tag{15}$$