Rutgers University – Physics Graduate Qualifying Exam
Electricity and Magnetism – August 27, 2008

Work problems A and B and (C1 or C2) and (D1 or D2).
Work each problem in a separate blue book.
Each problem is worth a total of 10 points.

EM - A

A charge q with mass m is released from rest at a distance D from an infinite conducting plane. How long will it take the charge to reach the plane?

Ignore gravity and radiation effects.

Useful integral: $\int_0^{\pi/2} \sin^2 y\, dy = \frac{\pi}{4}$

EM - B

A solid sphere of radius R centered at the origin is uniformly charged with charge density ρ. Then a small sphere is removed, making a spherical cavity centered at a point $\boldsymbol{a}$ within the sphere.

What is the electric field (magnitude and direction) inside the cavity?

EM - C1

A relativistic proton with a total energy in the laboratory frame (including rest energy) of $E_L = \gamma mc^2$ collides elastically with another proton that is initially at rest in the lab frame. After the collision both protons have the same total energy in the lab frame.

What is the angle between the final particles in the lab frame?

EM - C2

(a) [4 points] Consider an electromagnetic plane wave of angular frequency ω in a homogeneous, isotropic medium with electric permittivity, ε, and magnetic permeability, μ. Assume ε and μ are both positive.

Write down the spatial and temporal dependence of the fields **E**, **D**, **B**, and **H**.
Be sure to specify the <u>magnitude</u> and <u>direction</u> of all vectors, including that of **k**, in terms of ω, μ, ε, and the electric field amplitude E_o. Assume E_o is real and positive.

Show explicitly that these fields satisfy Maxwell's equations.

Compare the direction of the **k**-vector with that of the Poynting vector. (For simplicity, you may choose the propagation direction and polarization direction to be along two of the Cartesian axes.)

(b) [3 points] In what ways are the fields of part (a) modified in a medium in which the dielectric permittivity ε is real and negative, while the magnetic permeability μ is real and positive? Determine the instantaneous and time-averaged Poynting vectors. Can electromagnetic waves propagate in such a medium? Is there energy dissipation? Also consider the case $\varepsilon > 0$ and $\mu < 0$. Assume E_o is real and positive.

(c) [3 points] Repeat part (b) for a material in which both ε and μ are real and negative.
Can electromagnetic waves propagate in such a medium?
Is there energy dissipation?
Compare the direction of the **k**-vector (which characterizes motion of the wave fronts) with that of the Poynting vector (which characterizes energy flow). Assume E_o is real and positive.

EM - D1

(Note: You are not expected to provide a mathematically exact solution – instead, use physically reasonable and numerically accurate approximations.)

Consider a "split ring resonator", consisting of a circular conductive loop of diameter $4d$, made from a conductor of cross-sectional diameter d, in which a slit of width d has been left between the ends of the wire. The faces of the slit form the parallel plates of a wide-gap capacitor with a capacitance that is enhanced from the corresponding narrow-gap value by a fringe-field factor $\gamma \approx 2.3$. The ring may be considered a single winding of a solenoid, with inductance $L = \mu_0 r\left(\ln\frac{8r}{a} - 2\right)$, where $r = 2d$ is the radius of the loop and $a = d/2$ is the cross-sectional radius of the conductor. The conductor is a free electron metal with a conduction electron density of $n = 10^{23}/\mathrm{cm}^3$. (Electron mass $m_e = 9.1 \times 10^{-31}$ kg, electron charge $e = 1.6 \times 10^{-19}$ Coulomb.) Neglect electrical resistance and any retardation (radiation) effects.

(a) [2 points] What is the resonance frequency of the resulting LC circuit (neglecting the inertia of the electrons) as a function of diameter d?

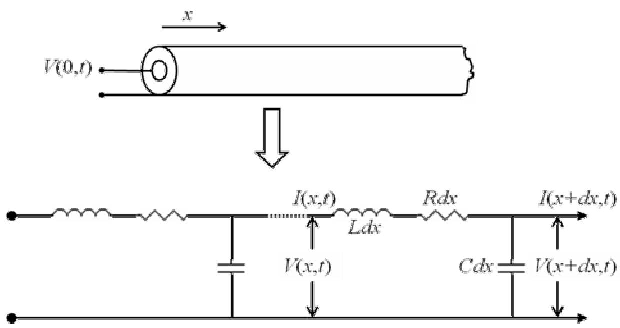
(b) [4 points] Now consider the case where the length scale d is made so small that the inductance L of the ring can be neglected (the so-called quasi-static limit). Including the effect of the mass (inertia) of the conduction electrons in the wire, the split ring again forms an electrical oscillator. The resulting oscillations are called plasmon modes. What is the frequency of the fundamental plasmon mode (in which all of the electrons are collectively displaced in a rigid rotation mode around the ring) as a function of d? If possible, give your answer in Hertz (s^{-1}).

(c) [4 points] Considering both inductance and inertia, what is the general expression for the resonance frequency as a function of d? For what values of d does the expression derived in part (a) apply, and for what values does the plasmon solution of part (b) apply? Give a numerical answer for the crossover length scale.

EM - D2

The coaxial cable is the most common transmission line used to carry signals at radio and microwave frequencies. The coaxial cable shown in the figure is composed of a center conductor of radius a, a coaxial cylindrical outer conductor of radius b, and an insulating dielectric material of permittivity, ε, in between. Assume that all the materials are non-magnetic. In other words, the permeability of the cable is equal to that of the free space, μ_0.

[In parts (c) and (d), you do not have to substitute for C and L the expressions obtained in parts (a) and (b).]



a) [2 points] Calculate the capacitance per unit length, C, for this coaxial cable.

b) [2 points] At high frequencies, the current through the center conductor is confined to the outer-most surface, and similarly current through the outer conductor is confined to the inner-most surface; in other words current I uniformly flows at radius a, and $-I$ at radius b. Under this condition, calculate the self inductance, L, per unit length.

c) [3 points] At high frequencies, the coaxial transmission line can be modeled as a series of inductors, resistors and capacitors as shown in the figure. Show that the current in the cable, $I(x,t)$, obeys an equation $\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + RC\frac{\partial I}{\partial t}$, where L, C, and R are self-inductance, capacitance, and resistance per unit length, respectively; Note that $V(x,t)$ also satisfies the same equational form.

d) [3 points] Suppose that a semi-infinite length ($x \geq 0$) of this cable is coupled at $x = 0$ to a sinusoidal signal generator of frequency ω so that $V(0,t) = V_0 \cos(\omega t)$. Assuming that the cable is lossless ($R = 0$), find the expression of the current $I(x,t)$ for x and $t > 0$ and show that the characteristic impedance of the cable $Z_c \equiv V(x,t)/I(x,t)$ is given by $\sqrt{L/C}$. Use the differential equations for I and V in part (c).