Solution CM-A

Using conservation of angular momentum and mechanical energy and considering the final state (primed) to where the rocket achieves maximum height, we have,

$v_r' = 0$

$mRv_\theta = m(R+H)v_\theta'$

$\frac{1}{2} m (v_\theta^2 + v_r^2) - GMm/R = \frac{1}{2} m v'^2_\theta - GMm/(R+H)$

where m is the mass of the rocket, and M is the mass of the Earth.

Combining the last two equations, we get:

$\frac{1}{2} m (v_\theta^2 + v_r^2) - GMm/R = \frac{1}{2} m [R/(R+H)]^2 v_\theta^2 - GMm/(R+H)$

which gives the maximum height H.

Considering only terms that are first order in H/R, we have:

$\frac{1}{2} m (v_\theta^2 + v_r^2) - GMm/R \approx \frac{1}{2} m (1-2H/R) v_\theta^2 - GMm/R (1-H/R)$

Solving for H, $H \approx v_r^2 R/[(2(GM/R - v_\theta^2)$.

For a vertical launch, $v_\theta = 0$, $v_r = \mathbf{v}$ and $H \approx v^2/2(GM/R^2) = v^2/2g$; the expected result.

Solution CM-B

Let N_z be the force exerted by the table to the moving part of the chain. The second Newton law for this part of the chain reads as follows,

$$\frac{\mathrm{d}}{\mathrm{d}t}(z\rho\dot{z}) = N_z + z\rho g \ ,$$

where z is a vertical coordinate of the end A and $\rho = \frac{M}{L}$. The impulse sustained by the table from the piece of the chain of length Δz is given by

$$\rho\dot{z}\Delta z = N_z \ \Delta t \quad \Longrightarrow \quad N_z = \rho\dot{z}^2$$

and, hence,

$$\ddot{z} = -g \quad \Longrightarrow \quad z(t) = L - \frac{gt^2}{2} \ .$$

The normal force exerted by the chain on the table is

$$R = N_z + (M - z\rho)g = \rho\dot{z}^2 + \rho(L - z)g \quad \Longrightarrow \quad R(t) = \frac{3}{2}\ \rho(gt)^2 \ .$$

The end A falls onto the table at the instant $\tau = \sqrt{\frac{2L}{g}}$, so that

$$R(\tau) = \frac{3}{2}\ \rho g^2\ \frac{2L}{g} = 3\,Mg \ .$$

Solution CM-C1

This is treated in detail in freshman textbooks, e.g. *Sears and Zemansky's University Physics* by Young and Friedman, volume 1, 12th edition, pages 498–504.

The power P crossing a point will be the energy per unit length (ε) at that point times the propagation speed ($v = \sqrt{T/\mu}$). This energy density ε is given by

$$\varepsilon = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = T\left(\frac{\partial y}{\partial x}\right)^2$$

with the final equality following because at finite times, x is replaced by $x - vt$. Carrying out the required derivative and substituting, one obtains for the power

$$P = \varepsilon v = \frac{4\xi^2 T v}{(1+\xi^2)^4}$$

where $\xi = (b - vt)/a$.

If your memory had forgotten the relationship between wave speed v and μ and T, and dimensional analysis had failed to refresh it, you may derive it as follows. Denoting partial t derivatives by an overdot and partial x derivatives by a prime, the Lagrangian is

$$L = \frac{1}{2}\int dx\,\left[\mu\dot{y}(x)^2 - Ty'(x)^2\right] = \frac{1}{2}\sum_k \left[\mu\,\dot{y}_k^*\dot{y}_k - T\,k^2 y_k^* y_k\right],$$

where y_k is the fourier representation of $y(x)$. The Lagrange equations are

$$\frac{d}{dt}\left(\frac{dL}{d\dot{y}_{k'}}\right) - \frac{dL}{dy_{k'}} = 0$$

and the complex conjugates. One obtains $\mu\ddot{y}_{k'} + Tk^2 y_{k'} = 0$ or $\mu\ddot{y}(x) - Ty''(x) = 0$, i.e. the equation for a wave moving with the velocity v as given above. A more cumbersome derivation based on Newton's laws is given in the freshman textbook mentioned above.

Solution CM-C2

a) Equations of motion:

$m_1 [d^2r/dt^2 - r\,(d\theta/dt)^2] = -T$ (1)

$m_1 r^2\, d\theta/dt = m_1 h$ (2)

$T - m_2 g = m_2\, d^2r/dt^2$ (3)

Where $m_1 h$ is the constant angular momentum.

Eliminate T:
$(m_1 + m_2) d^2r/dt^2 - m_1 r\,(d\theta/dt)^2 + m_2 g = 0$ (4)

Using (2) and (4) to eliminate θ,

$(m_1 + m_2) d^2r/dt^2 - m_1 h^2/r^3 = -m_2 g$. (5)

Since $d^2r/dt^2 = dr/dt \cdot d(dr/dt)/dr = ½\, d\,(dr/dt)^2 / dr$, we can now integrate (5)

$½\,(m_1 + m_2)\,(dr/dt)^2 + m_1 h^2/2r^2 = -m_2 g r + C$ (6)

At t= 0, $r = \mathbf{R}_0$ and $dr/dt = V_0 \cos\varphi$ $r\, d\theta/dt = V_0 \sin\varphi$, where φ is the angle between $\mathbf{R}_0$ and $\mathbf{V}_0$. Thus,

$h = R_0 V_0 \sin\varphi$, and

$C = ½\, [(m_1 + m_2)\, V_0^2 \cos^2\varphi$ and $m_1 V_0^2 \sin^2\varphi] + m_2 g R_0$

For r to be an extremum, dr/dt=0, and (6) becomes:

$2m_2 g r^3 - 2Cr^2 + m_1 h^2 = 0$, whose solutions give the maximum and minimum radial distances.

b) When the orbit is circular, $d^2r/dt^2 = 0$ and (5) gives:

$h^2 = m_2 g r_0^3/m_1$, where r_0 is the radius of the circular orbit.
For small deviations, let $r = r_0 + x$, where $x << r_0$. Now, (5) becomes:

$(m_1 + m_2) d^2x/dt^2 - m_1 h^2 / (r_0 + x)^3 = -m_2 g$.

$(r_0 + x)^{-3} = r_0^{-3} (1 + x/r_0)^{-3} \approx r_0^{-3} (1-3x/r_0)$,

So,

$(m_1 + m_2)d^2x/dt^2 - m_1 h^2 / (r_0^{-3} - 3xr_0^{-4}) = -m_2g.$

Substituting for h,

$(m_1 + m_2)d^2x/dt^2 + 3m_2gx/r_0 = 0.$

This is SHO, with $\omega/2\pi = 1/2\pi \sqrt{} [3m_2g/(m_1 + m_2)r_0]$

Solution CM-D1

Let Ωt be the angle between the diameter going through the rotation point O and the x axis. Then we may write the cartesian coordinates of the mass as

$$\begin{aligned} x &= b\cos\Omega t + b\cos(\theta + \Omega t) \\ y &= b\sin\Omega t + b\sin(\theta + \Omega t) \end{aligned}$$

The kinetic energy is then

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{mb^2}{2}\left[\Omega^2 + (\Omega+\omega)^2 + 2\Omega(\Omega+\omega)\cos\theta\right],$$

where $\omega = \dot{\theta}$. Applying the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{dT}{d\theta} = 0$$

gives [$\omega = \dot{\theta}$]

$$\ddot{\theta} + \Omega^2\sin\theta = 0,$$

which is the required equation. This is the same as the equation for a simple pendulum of length g/Ω^2. From this we conclude that the motion is oscillatory for $|\omega_0| < 2\Omega$ and circular for $|\omega_0| > 2\Omega$. The period $T = 2\pi/\Omega$ for $|\omega_0| \ll \Omega$ and increases, becoming very long when $|\omega_0| \sim 2\Omega$. For $|\omega_0| > 2\Omega$ the period decreases again with increasing $|\omega_0|$, and approaches $2\pi/|\omega_0|$ for $|\omega_0| \gg 2\Omega$.

To find the force of constraint, we allow a virtual displacement of the mass in a direction perpendicular to the wire. If we let this displacement be $r-b$, then the cartesian coordinates of the mass become

$$\begin{aligned} x &= b\cos\Omega t + r\cos(\theta + \Omega t) \\ y &= b\sin\Omega t + r\sin(\theta + \Omega t) \end{aligned}$$

The kinetic energy is then

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left[b^2\Omega^2 + \dot{r}^2 + r^2(\Omega+\omega)^2 + 2b\Omega\dot{r}\sin\theta + 2br\Omega(\Omega+\omega)\cos\theta\right],$$

The Lagrange equation for the r degree of freedom is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{r}}\right) - \frac{dT}{dr} = F_r,$$

where F_r is the force of constraint to be determined. Taking the required derivatives of T and then setting $r = \text{constant} = b$ gives the desired result

$$F_r = -mb\left[\Omega^2\cos\theta + (\dot{\theta}+\Omega)^2\right].$$

Solution CM-D2

Substitution the ansatz $u(x,t) = y(\tau)$ with $\tau = x - vt$ into the KdV equation leads to the ordinary differential equation

$$-v \, \frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{\mathrm{d}^3 y}{\mathrm{d}\tau^3} + 6y \, \frac{\mathrm{d}y}{\mathrm{d}\tau} = 0 \ ,$$

or, integrating with respect to τ,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = -A + v\, y - 3y^2 \ ,$$

where A is a constant of integration. Interpreting the independent variable τ above as a time variable, this means y satisfies Newton's equation of motion in a cubic potential

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}U}{\mathrm{d}y} \ , \qquad \text{where} \qquad U(y) = Ay - \frac{v}{2}\, y^2 + y^3 \ .$$

The energy conservation law for this one-dimensional motion gives

$$E = \frac{1}{2} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 + U(y) \ .$$

Depending on the value of the real constants A and v the cubic polynomial $U(y)$ may have one or tree real roots. We are interesting in finding of nontrivial ($y \neq 0$) *finite* motions such that $y(\tau) < const$ as $\tau \to \pm\infty$. The motion is finite if $U(y)$ has three real roots and

$$U_{min} \leq E \leq U_{max} \ .$$

Otherwise the motion is *infinite*. Therefore there are three possible types of automodel solutions:

- Constant solution, $u(x,t) = const$:

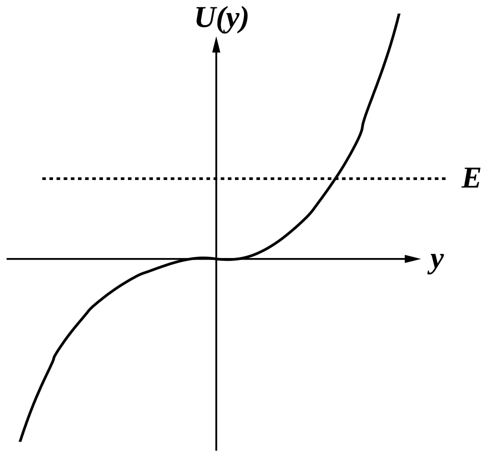
$$E = U_{min} \ , \quad y = y_{min} = const \ .$$

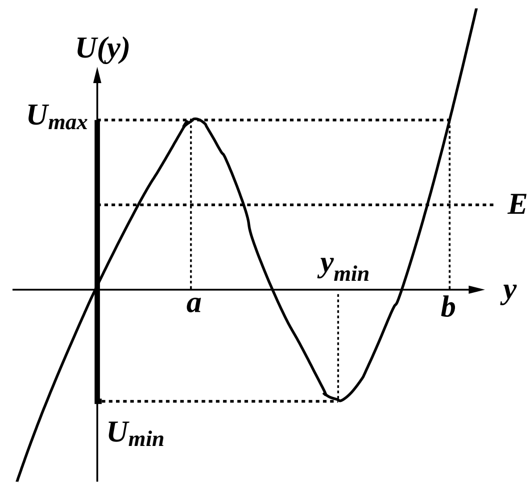
- The so called cnoidal waves corresponding to the periodic motions with

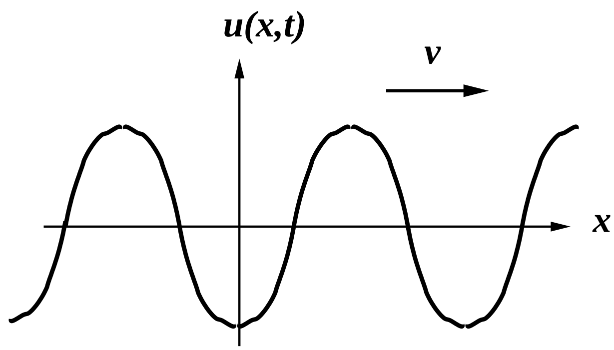
$$U_{min} < E < U_{max} \ .$$

- Finite motion with

$$E = U_{max} \ .$$







In this case the profile $y(\tau)$ starts at $y = y_{\rm max} = a$ at "time" $\tau \to -\infty$, eventually slides down to the local minimum, then back to the other side, reaching at $y = b$ an equal height $U_{\rm max}$, then reverse direction, ending up at $y = y_{\rm max} = a$ again at "time" $\tau = +\infty$.

For this motion

$$\frac{1}{2}\left(\frac{{\rm d}y}{{\rm d}\tau}\right)^2 = U_{\rm max} - U(y) = (y-a)^2\ (b-y)\ , \qquad \text{where} \qquad b = \frac{v}{2} - 2a,$$

and, hence,

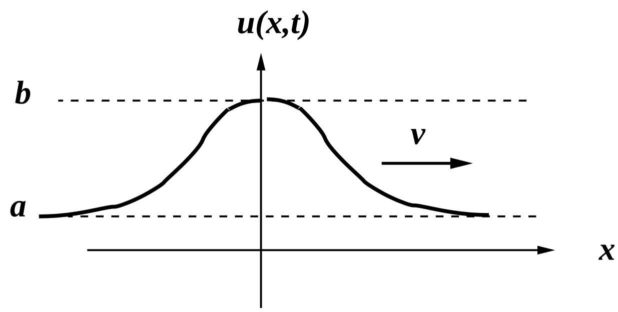
$$\pm \int \frac{{\rm d}y}{(y-a)\sqrt{2(b-y)}} = \tau\ .$$

Changing the integration variable

$$y = a + (b-a)z = a + \frac{b-a}{\cosh^2(\frac{\theta}{2})}$$

one obtains

$$\int \frac{{\rm d}z}{z\sqrt{1-z}} = -\theta = \pm\sqrt{2(b-a)}\ \tau\ ,$$



or

$$y(\tau) = a + \frac{b-a}{\cosh^2\left(\sqrt{\frac{b-a}{2}}\ (\tau - \tau_0)\right)} \ .$$

If the parameters are adjusted so that $a = y_{max} = 0$, then $y(\tau)$ approaches to 0 as $\tau \to \pm\infty$. In this case

$$u_{sol}(x,t) = y_{sol}(x - vt) = \frac{v}{2\cosh^2\left(\frac{\sqrt{v}}{2}\ (x - vt - x_0)\right)} \ .$$

It describes a right-moving localized wave packet, i.e., soliton.