Rutgers University – Physics Graduate Qualifying Exam
Classical Mechanics – August 27, 2008

Work problems A and B and (C1 or C2) and (D1 or D2).
Work each problem in a separate blue book.
Each problem is worth a total of 10 points.

CM - A

A rocket is fired from the surface of the Earth (radius R) with velocity $\mathbf{v} = (v_r, v_\theta)$.

Neglecting air resistance and rotation of the Earth, obtain an equation to determine the maximum height H achieved by the trajectory. Solve it to lowest order in (H/R) and verify that it gives a familiar result for the case that $\mathbf{v}$ is vertical.

CM - B

A flexible uniform chain AB has length L and mass M. It is suspended vertically over a horizontal table with its end B just touching the table, and it released so that it collapses inelastically onto the table. Calculate the normal force exerted on the table the instant when the end A falls onto the table.

CM - C1

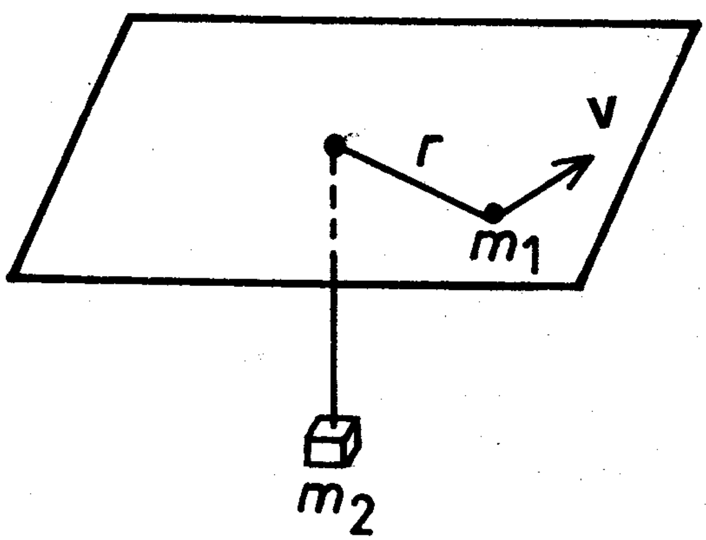
Consider the motion of a transverse mechanical wave along a long wire kept under a tension T and whose mass per unit length is μ. Assume that the wire runs in the x-direction and that the only displacements are in the y-direction. Assume that these displacements are small. At time t = 0, the wave function is given by:

$$y(x) = \frac{a^3}{a^2 + x^2}$$

Where a is a constant. Assume that this function contains only components moving in the positive x-direction. Find the power that crosses the point $x = b$ as a function of time t.

CM - C2

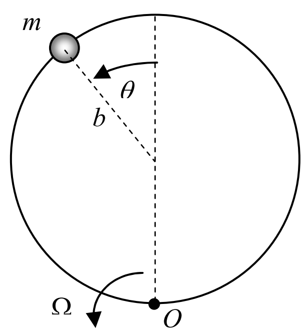
A mass m_1 moves around a hole on a frictionless horizontal table. The mass is tied to a string which passes through the hole. A mass m_2 is tied to the other end of the string.



a) [5 points] Given the initial position $\mathbf{R}_0$ and $\mathbf{v}_0$ in the plane of table and masses m_1 and m_2, find the equation that determines maximum and minimum radial distances of the orbit. (Do NOT solve it!)
b) [5 points] Find the frequency of oscillation of the radius of the orbit when the orbit is only slightly different from circular.

CM - D1

Consider a bead of mass m sliding freely on a smooth circular wire of radius b which rotates in a horizontal plane about one of its points O, with constant angular velocity Ω. Let θ be the counterclockwise angle between the diameter that passes through the mass and the diameter that passes through the point O, with $\theta = 0$ the case where the mass is farthest from O.



(a) [5 points] Find the differential equation that determines θ. Letting ω_o be the value of $\dot{\theta}$ when $\theta = 0$, describe in words the θ motion that occurs for different ranges of ω_o.

(b) [5 points] Find the force that the wire exerts on the bead as a function of θ and $\dot{\theta}$.

CM - D2

The Korteweg-de Vries equation is a mathematical model of waves on shallow water surface:

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}=0 \quad \text{where } u = u(x,\, t)$$

This partial differential equation possesses automodel solutions of the form

$$u(x,\, t) = y(\tau) \text{ with } \tau = x - vt$$

and $y(\tau)$ is a function bounded as $\tau \to \pm\infty$, *i.e.*, $|\, u(x,t)\,|$ < const. as $x \to \pm\infty$ for any instant t. The automodel solutions describe wave packets traveling with the constant speed v without dispersion.

(a) [3 points] Reduce the problem of finding of the automodel solutions to the problem of motion of particle in some conservative field in one dimension. Show that the energy can be written as:

$$E=\frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2+U(y)$$

where $U(y)$ is a cubic polynomial in y. Find the specific form of the polynomial $U(y)$, giving the values of those coefficients which are determined by the original partial differential equation.

(b) [3 points] Suppose the polynomial U has three real roots, so that there will be a local maximum, $U_{\max}$, and a local minimum, $\underline{U}_{\min}$ straddling the middle root, and which are in turn straddled by the largest and smallest root. Discuss the nature fo the solution for the cases $E = U_{\min}$, $U_{\min} < E < U_{\max}$, $E > U_{\max}$. Which one of these satisfy the requirement that $|\, u(x,t)\,|$ < const. as $x \to \pm\infty$?

(c) [4 points] Show that among the automodel solutions there is the so called solitary wave (soliton) describing a *localized* wave packet whose profile $y_{\mathrm{sol}}(\tau)$ approaches to 0 as $\tau \to \pm\infty$. Find an explicit form of the soliton.

Useful integral:

$$\int \frac{dz}{z\sqrt{1-z}} = -2\,\mathrm{arccosh}\left(\frac{1}{\sqrt{z}}\right)$$